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# SIMULTANEOUS EXTENSIONS OF PROXIMITIES, SEMI-UNIFORMITIES, CONTIGUITIES AND MEROTOPIES III*

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**Abstract:** Given compatible merotopies (or contiguities) on some subspaces of a proximity space, we are looking for a common extension of these structures.

§§ 0 and 1 can be found in Part I [1], §§ 2 to 4 in Part II [2]. See § 0 for terminology, notations and conventions. We shall also need the following notations introduced later: $A^r = X \setminus A$ (for $A \subset X$); $\mathsf{M}^0(\Gamma)$ is the merotopy for which the contiguity $\Gamma$ constitutes a base (cf. 4.1).

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# 5. Extending a family of merotopies in a proximity space

## A. WITHOUT SEPARATION AXIOMS

**5.1** A family of merotopies in a proximity space always has an extension; we are going to construct the coarsest one. In general, there is no finest extension, not even for $I = \emptyset$; this could be deduced from the well-known fact that there may fail to exist a finest compatible uniformity in an Efremovich proximity space (see e.g. [5] Ch. I, Ex. 12.), but we shall give a simpler example in 5.3.

**Definition.** A cover $\mathsf{c}$ in a proximity space $(X, \delta)$ is a *$\delta$-cover* if $A\delta B$ implies the existence of a $C \in \mathsf{c}$ with $A \cap C \neq \emptyset \neq B \cap C$. $\diamondsuit$

In other words, $\mathsf{c}$ is a $\delta$-cover iff for any $A \subset X$, $A\bar{\delta}\mathrm{St}(A, \mathsf{c})^r$. Evidently, any cover refined by some $\delta$-cover is a $\delta$-cover.

**Lemma.** *For a merotopy* $\mathsf{M}$ *on* $X$, $\delta(\mathsf{M})$ *is coarser than* $\delta$ *iff every* $\mathsf{c} \in \mathsf{M}$ *is a $\delta$-cover iff* $\mathsf{M}$ *has a base consisting of $\delta$-covers.*

**Proof.** 0.4 (1). $\diamondsuit$

**5.2 Notation.** For $\mathsf{a} \subset \exp X$, Let $\mathsf{pa}$ denote the partition of $X$ generated by $\mathsf{a}$; this means that $S \in \mathsf{pa}$ iff $S = \bigcap_{A \in \mathsf{a}} f(A)$, where, for each $A \in \mathsf{a}$, either $f(A) = A$ or $f(A) = A^r$. $\diamondsuit$

**Lemma.** *If* $\mathsf{c}$ *and* $\mathsf{f}$ *are $\delta$-covers, and* $\mathsf{f}$ *is finite then* $\mathsf{c}(\cap)\mathsf{f}$ *is a $\delta$-cover as well.*

**Proof.** By Axiom P5, we may assume when checking the condition in Definition 5.1 that there are $A'$, $B' \in \mathsf{pf}$ with $A \subset A'$, $B \subset B'$. As $\mathsf{f}$ is a $\delta$-cover, there is a $D \in \mathsf{f}$ such that $A \cup B \subset A' \cup B' \subset D$. $\mathsf{c}$ is also a $\delta$-cover, so we can pick a $C \in \mathsf{c}$ with $A \cap C \neq \emptyset \neq B \cap C$. Now $C \cap D \in \mathsf{c}(\cap)\mathsf{f}$, and $A \cap (C \cap D) \neq \emptyset \neq B \cap (C \cap D)$. $\diamondsuit$

It is not superfluous to assume that $\mathsf{f}$ is finite:

**Example.** Let $X = \mathbb{N}$, $P = \{2n : n \in \mathbb{N}\}$, $Q = P^r$. For disjoint $A, B \subset X$, let $A\delta B$ iff both $A$ and $B$ are infinite. Now

$$\mathsf{c} = \big\{\{p, q\} : p \in P,\ q \in Q,\ p < q\big\} \cup \{P, Q\}$$

and $\mathsf{d}$ defined analogously, with $p > q$ substituted for $p < q$, are $\delta$-covers, but $\mathsf{c}(\cap)\mathsf{d}$ is not a $\delta$-cover. $\diamondsuit$

**5.3 Definition.** For a family of merotopies in a proximity space, let

$\mathsf{M}^0$ be the merotopy for which the following covers form a subspace $\mathsf{B}$:

$$\mathsf{c}_i^0 = \{C_i^0 = C_i \cup X_i^r : C_i \in \mathsf{c}_i\} \quad (i \in I, \mathsf{c}_i \in \mathsf{M}_i);$$
$$\mathsf{c}_{A,B} = \{A^r, B^r\} \quad (A\bar{\delta}B). \ \diamondsuit$$

Recall the covers $\mathsf{c}_i^0$ were already introduced in §3. We shall write $\mathsf{M}^0(\delta, \mathsf{M}_i) = \mathsf{M}^0(\delta, \{\mathsf{M}_i : i \in I\})$ when necessary, e.g. when it has to be distinguished from $\mathsf{M}^0(c, \mathsf{M}_i)$. $\mathsf{M}^0(\delta) = \mathsf{M}^0(\delta, \emptyset)$.

**Lemma.** *Let $(X, \delta)$ be a proximity space.*

a) *For $A\bar{\delta}B$, $\mathsf{c}_{A,B}$ is a $\delta$-cover.*

b) *$\mathsf{M}^0(\delta)$ is the coarsest merotopy compatible with $\delta$.*

c) *For $X_0 \subset X$, $\mathsf{M}^0(\delta)|X_0 = \mathsf{M}^0(\delta|X_0)$.*

d) *A filter on $X$ is $\mathsf{M}^0(\delta)$-Cauchy iff it is $\delta$-compressed.*

**Proof.** a) If $\emptyset \neq E \subset A$ then $\mathrm{St}(E, \mathsf{c}_{A,B})^r = B$ and $E\bar{\delta}B$; the case $\emptyset \neq E \subset B$ is analogous; finally, if $E \not\subset A$, $E \not\subset B$ then $\mathrm{St}(E, \mathsf{c}_{A,B}) = X$. Thus $\mathsf{c}_{A,B}$ satisfies the condition mentioned after Definition 5.1.

b) By Lemmas 5.2 and 5.1, $\delta(\mathsf{M}^0(\delta))$ is coarser than $\delta$. Conversely, if $A\bar{\delta}B$ then $\mathrm{St}(A, \mathsf{c}_{A,B}) \cap B = \emptyset$, thus $A\bar{\delta}(\mathsf{M}^0(\delta))B$. Hence $\mathsf{M}^0(\delta)$ is compatible.

If $\mathsf{M}$ is compatible and $A\bar{\delta}B$ then there is a $\mathsf{c} \in \mathsf{M}$ such that $\mathrm{St}(A, \mathsf{c}) \cap B = \emptyset$. Now $\mathsf{c}$ refines $\mathsf{c}_{A,B}$, so $\mathsf{c}_{A,B} \in \mathsf{M}$ and $\mathsf{M}^0(\delta) \subset \mathsf{M}$.

c) Clearly

$$\mathsf{c}_{A,B}|X_0 = \mathsf{c}_{A\cap X_0, B\cap X_0},$$

with the right hand side understood in the fundamental set $X_0$, and $A\bar{\delta}B$ implies $A \cap X_0 \bar{\delta}_0 B \cap X_0$, while if $A\bar{\delta}_0 B$ then $A\bar{\delta}B$ (where $\delta_0 = \delta|X_0$).

d) Recall that a filter is Cauchy iff it intersects each elements of a given subbase. $\diamondsuit$

There is, in general, no finest compatible merotopy:

**Example.** Take $(X, \delta)$, $\mathsf{c}$ and $\mathsf{d}$ from Example 5.2. By Lemmas 5.1, 5.2 and 5.3 b), $M^0(\delta) \cup \{\mathsf{c}\}$ and $M^0(\delta) \cup \{\mathsf{d}\}$ are subbases for compatible merotopies. A finest compatible merotopy would have to contain $\mathsf{c}(\cap)\mathsf{d}$, which is not a $\delta$-cover. $\diamondsuit$

The induced closure is discrete in this example, thus any merotopy compatible with $\delta$ is Lodato. Consequently, there does not exist a finest compatible Lodato (or Riesz) merotopy.

**5.4 Theorem.** *A family of merotopies in a proximity space can always* *be extended;* $\mathsf{M}^0$ *is the coarsest extension.*

**Proof.** 1° $\delta(\mathsf{M}^0)$ *is finer than* $\delta$. This follows from $\mathsf{M}^0(\delta) \subset \mathsf{M}^0$ and Lemma 5.3 b).

2° $\delta(\mathsf{M}^0)$ *is coarser than* $\delta$. It is enough to show that if $\emptyset \neq F \subset I$ is finite and $\mathbf{c}_i \in \mathsf{M}_i$ ($i \in F$) then $\mathbf{c} = (\bigcap_{i\in F})\mathbf{c}_i^0$ is a $\delta$-cover, since $\mathbf{c}_{A,B}$ is a $\delta$-cover by Lemma 5.3 a), so Lemma 5.2 yields that the elements of $\mathsf{M}^0$ are $\delta$-covers, and then Lemma 5.1 can be applied.

Let $A\delta B$; a $C \in \mathbf{c}$ with $A \cap C \neq \emptyset \neq B \cap C$ is needed. By Axiom P5, we may assume that there are $A'$, $B' \in \mathbf{p}\{X_i : i \in F\}$ such that $A \subset A'$, $B \subset B'$. Let us decompose the index set $F$ into four parts as follows:

$$\begin{array}{cc} A \cup B \subset X_i & (i \in F_0); \\ A \subset X_i,\ B \subset X_i^\tau & (i \in F_1); \\ A \subset X_i^\tau,\ B \subset X_i & (i \in F_2); \\ A \cup B \subset X_i^\tau & (i \in F_3). \end{array}$$

By the accordance, $\mathsf{M}_i|A \cup B$ is the same merotopy compatible with $\delta|A \cup B$ for each $i \in F_0$, and $(\bigcap_{i\in F_0})(\mathbf{c}_i|A \cup B)$ belongs to it, so we can choose $C_i \in \mathbf{c}_i$ ($i \in F_0$) such that

$$A \cap \bigcap_{i\in F_0} C_i \neq \emptyset \neq B \cap \bigcap_{i\in F_0} C_i. \tag{1}$$

Fix now points $x$ and $y$ from the left hand side, respectively the right hand side of (1); in case $F_0 = \emptyset$, assume only that $x \in A$, $y \in B$. For $i \in F_1$, pick $C_i \in \mathbf{c}_i$ with $x \in C_i$; similarly, for $i \in F_2$, let $y \in C_i \in \mathbf{c}_i$. For $i \in F_3$, take an arbitrary set $C_i \in \mathbf{c}_i$. With $C = \bigcap_{i\in F} C_i^0 \in \mathbf{c}$ we have $x \in A \cap C$, $y \in B \cap C$.

3° $\mathsf{M}^0|X_i$ *is finer than* $\mathsf{M}_i$, since for any $\mathbf{c}_i \in \mathsf{M}_i$, $\mathbf{c}_i^0|X_i = \mathbf{c}_i$, and $\mathbf{c}_i^0 \in \mathsf{M}^0$.

4° $\mathsf{M}^0|X_i$ *is coarser than* $\mathsf{M}_i$. By Lemma 5.3 c), $\mathbf{c}_{A,B}|X_i \in \mathsf{M}^0(\delta_i)$, so Lemma 5.3 b) implies that it belongs to $\mathsf{M}_i$. $\mathbf{c}_j^0|X_i \in \mathsf{M}_i$ follows from the accordance: Taking a $\mathbf{c}_i \in \mathsf{M}_i$ with $\mathbf{c}_i|X_{ij} = \mathbf{c}_j|X_{ij}$, $\mathbf{c}_i$ will refine $\mathbf{c}_j^0|X_j$, since if $C_i \in \mathbf{c}_i$ then $C_i \cap X_{ij} = C_j \cap X_{ij} = C_j^0 \cap X_{ij}$ for some $C_j \in \mathbf{c}_j$, and $C_i \subset (C_j^0 \cap X_{ij}) \cup (X_i \setminus X_{ij}) = C_j^0 \cap X_i$.

5° $\mathsf{M}^0$ *is the coarsest extension.* Let $\mathsf{M}$ be another extension. $\mathbf{c}_{A,B} \in \mathsf{M}$ by Lemma 5.3 b). For $\mathbf{c}_i \in \mathsf{M}_i$, take a $\mathbf{c} \in \mathsf{M}$ with $\mathbf{c}_i = \mathbf{c}|X_i$; now $\mathbf{c}$ refines $\mathbf{c}_i^0$, thus $\mathbf{c}_i^0 \in \mathsf{M}$, too. Hence $\mathsf{M}^0 \subset \mathsf{M}$. ◇

**5.5 Theorem.** *A family of merotopies in a proximity space has a finest exension iff* $\mathsf{c}(\cap)c'$ *is a* $\delta$*-cover whenever* $\mathsf{c}$ *and* $\mathsf{c}'$ *are* $\delta$*-covers with traces belonging to* $\mathsf{M}_i$ $(i \in I)$*. If so then these covers make up the finest extension* $\mathsf{M}^1$.

**Proof.** Any cover belonginng to an extension is a $\delta$-cover with traces in $\mathsf{M}_i$, so if the system of these covers is closed for the operation $(\cap)$ then they constitute a merotopy finer than each extension, and this merotopy is an extension by Lemma 5.1 and Theorem 5.4.

Conversely, assume that there exists a finest extension $\mathsf{M}^1$. If $\mathsf{c} \in \mathsf{M}^1$ then $\mathsf{c}$ is a $\delta$-cover by Lemma 5.1; $\mathsf{c}|X_i \in \mathsf{M}_i$ is evident. If $\mathsf{d}$ is a $\delta$-cover and $\mathsf{d}|X_i \in \mathsf{M}_i$ $(i \in I)$ then $\mathsf{M}^0 \cup \{\mathsf{d}\}$ is a subbase for an extension $\mathsf{M}$. $\mathsf{M}|X_i = \mathsf{M}_i$ is clear; $\mathsf{M}$ is compatible, as $\mathsf{M}^0 \subset \mathsf{M}$ and the elements of $\mathsf{M}$ are $\delta$-covers; the last statement can be proved using the argument from 2° of the proof of Theorem 5.4, with the changement that $\mathsf{d}|A \cup B$ has to be added to the covers $\mathsf{c}_i|A \cup B$ $(i \in F_0)$, thus $\mathsf{d} \in \mathsf{M}^1$. Hence $\mathsf{M}^1$ consists of the $\delta$-covers with traces in $\mathsf{M}_i$. ◇

**5.6** For a non-empty family of merotopies in a proximity space, we have

$$\mathsf{M}^0 = \sup_{i\in I} \mathsf{M}^0(\delta, \{\mathsf{M}_i\}) = \sup\{\mathsf{M}^0(\delta), \sup_{i\in I} \mathsf{M}^{00}[i]\}, \tag{1}$$

where $\mathsf{M}^{00}[i]$ is the coarsest merotopy $\mathsf{M}$ on $X$ for which $\mathsf{M}|X_i = \mathsf{M}_i$, i.e. $\{\mathsf{c}_i^0 : \mathsf{c}_i \in \mathsf{M}_i\}$ is a base for $\mathsf{M}^{00}[i]$. (1) follows from 2.2 a), but can also be easily seen from Definition 5.3. (Recall that for merotopies $\mathsf{M}[i]$ $(i \in I \neq \emptyset)$ on $X$, $\bigcup_{i\in I} \mathsf{M}[i]$ is a subbase for $\sup_{i\in I} \mathsf{M}[i]$.)

**5.7** A part of Theorem 3.1 can be deduced in two steps from Theorems 1.2 and 5.4: given a family of merotopies in a symmetric closure space, extend first the induced proximities, and then take the merotopy $\mathsf{M}(\delta^0, \mathsf{M}_i)$; this merotopy is the coarsest extension in $(X, c)$: if $\mathsf{M}$ is another extension then $\delta(\mathsf{M})$ is an extension of the proximities $\delta(M_i)$, thus it is finer then $\delta^0$; now

$$\mathsf{M}^0(\delta^0, \mathsf{M}_i) \subset \mathsf{M}^0(\delta(\mathsf{M}), \mathsf{M}_i) \subset \mathsf{M}$$

(the first inclusion can be seen from Definition 5.3, the second one follows from Theorem 5.4, since $\mathsf{M}$ is an extension in the proximity space $(X, \delta(\mathsf{M}))$. Therefore:

$$\mathsf{M}^0(c, \mathsf{M}_i) = \mathsf{M}^0(\delta^0(c, \delta(\mathsf{M}_i)), \mathsf{M}_i). \tag{1}$$

If we only want to prove the *existence* of an extension of a family of merotopies in a closure space then $\delta^1$ can also be used instead on $\delta^0$, but $\mathsf{M}^0(\delta^1, \mathsf{M}_i)$ is in general different from $\mathsf{M}^1(c, \mathsf{M}_i)$. It is, however, true that $\mathsf{M}^1(c, \mathsf{M}_i)$ is the finest extension of the merotopies in $(X, \delta^1)$ (because it is an extension in $(X, c)$ finer than $\mathsf{M}^0(\delta^1, \mathsf{M}_i)$, so it induces a proximitiy $\delta'$ finer that $\delta^1$; $\delta'$ is an extension of the proximities $\delta(M_i)$, so it is also coarser than $\delta^1$; thus $\mathsf{M}^1(c, \mathsf{M}_i)$ is indeed an extension in $(X, \delta^1)$, and it is the finest one in a larger class of merotopies, namely the extensions in $(X, c)$). Therefore:

$$\mathsf{M}^1(c, \mathsf{M}_i) = \mathsf{M}^1(\delta^1(c, \delta(\mathsf{M}_i)), \mathsf{M}_i). \tag{2}$$

But there arises a difficulty if we try to deduce the part of Theorem 3.1 concerning finest extensions: it has to be shown somehow that Theorem 5.5 applies to $\delta^1$.

**5.8** Conversely, it is also possible to base the proof of Theorems 1.1 and 1.2 on Theorem 3.1 and Lemma 5.3:

Let a family of proximities be given in a symmetric closure space. By Lemma 5.3 b) and c), $\{\mathsf{M}^0(\delta_i) : i \in I\}$ is a family of merotopies in $(X, c)$; Theorem 3.1 furnishes the coarsest, respectively the finest extension $\mathsf{M}^0$ and $\mathsf{M}^1$ of this family. Now $\delta(\mathsf{M}^0)$ and $\delta(\mathsf{M}^1)$ are clearly extensions of the family of proximities. If $\delta$ is an extension of the same proximities then $\mathsf{M}^0(\delta)$ is an extension of the merotopies $\mathsf{M}^0(\delta_i)$ (again by Lemma 5.3 c)), thus $\mathsf{M}^0 \subset \mathsf{M}^0(\delta) \subset \mathsf{M}^1$, implying $\delta(\mathsf{M}^0) \supset \delta \supset \supset \delta(\mathsf{M}^1)$. So $\delta(M^0)$ and $\delta(M^1)$ are coarsest, respectively finest. Therefore we have:

$$\delta^k(c, \delta_i) = \delta(\mathsf{M}^k(c, \mathsf{M}^0(\delta_i))) \qquad (k = 0, 1). \tag{1}$$

(Compare these formulas with 4.1 (1).)

## B. RIESZ MEROTOPIES IN A PROXIMITY SPACE

**5.9 Theorem.** *A family of merotopies in a proximity space has a Riesz extension iff the proximity is Riesz and the trace filters are Cauchy; if so then* $\mathsf{M}^0$ *is the coarsest Riesz extension.*

**Proof.** The conditions are clearly necessary. Conversely, if they are satisfied then $\mathsf{M}^0$ is Riesz (so it is the coarsest Riesz extension by Theorem 5.4):

Let $x \in X$ and $\mathsf{c} \in \mathsf{B}$ (see Definition 5.3) be fixed; we need a $C \in \mathsf{c}$ with $x \in \operatorname{int} C$. If $\mathsf{c} = \mathsf{c}_{A,B}$, $A\bar{\delta}B$ then $x \notin c(A)$ or $x \notin c(B)$ (as $\delta$ is Riesz), thus $x \in \operatorname{int} A^r$, $A^r \in \mathsf{c}$, or $x \in \operatorname{int} B^r$, $B^r \in \mathsf{c}$. If $\mathsf{c} = \mathsf{c}_i^0$, $i \in I$, $\mathsf{c}_i \in \mathsf{M}_i$ then there is a $C_i \in \mathsf{c}_i \cap \mathsf{s}_i(x)$ (as the trace filters are Cauchy), thus $C_i^0 \in \mathsf{v}(x)$, i.e. $x \in \operatorname{int} C_i^0$, $C_i^0 \in \mathsf{c}$. ◇

**5.10 Theorem.** *A family of merotopies in a proximity space has a finest Riesz extension iff $\delta$ is Riesz, the trace filters are Cauchy, and* $\mathsf{c}(\cap)\mathsf{c}'$ *is a $\delta$-cover whenever* $\mathsf{c}$ *and* $\mathsf{c}'$ *are $\delta$-covers with traces belonging to* $\mathsf{M}_i$ $(i \in I)$ *such that* $\operatorname{int}\mathsf{c}$ *and* $\operatorname{int}\mathsf{c}'$ *are covers. If so then these covers make up the finest Riesz extension* $\mathsf{M}_R^1$.

**Proof.** If $\mathsf{M}_R^1$ exists then $\mathsf{M}^0$ is Riesz by Theorem 5.9. Now assuming in the proof of Theorem 5.5 that $\operatorname{int}\mathsf{d}$ is a cover, the extension $\mathsf{M}$ defined there is Riesz, thus $\mathsf{d} \in \mathsf{M}_R^1$. ◇

If the conditions of Theorem 5.9 are fullfilled and there exists a finest extension $\mathsf{M}^1$ then so does $\mathsf{M}_R^1$ (take those $\mathsf{c} \in \mathsf{M}^1$ for which $\operatorname{int}\mathsf{c}$ is a cover), but not conversely, not even for $I = \emptyset$:

**Example.** Take $X = [-1,1]$ with the Euclidean proximity $\delta$. Let

$$\mathsf{c} = \{[-1,0],\, [0,1]\} \cup \{\{p,q\} : 0 < -p < q < 1\},$$

and $\mathsf{d}$ defined analogously, with $0 < q < -p < 1$. $\mathsf{c}$ and $\mathsf{d}$ are $\delta$-covers, but $\mathsf{c}(\cap)\mathsf{d}$ is not a $\delta$-cover, so (as in Example 5.3) there is no finest compatible merotopy. But there exists a finest compatible Riesz merotopy, namely the one for which all the open covers form a base. ◇

**5.11** It can also occur that $\mathsf{M}^1$ and $\mathsf{M}_R^1$ both exist but differ: let $\delta$ be the indiscrete proximity on a three-point set. A better example, with $\delta$ separated:

**Example.** Let $X$ be infinite, $z \in X$, and $\mathsf{u}$ a free ultrafilter on $X$. Take the topology $c$ on $X$ for which $\big\{\{z\} \cup S : S \in \mathsf{u}\big\}$ is the neighbourhood filter of $z$, and the other points are isolated. Now with $\delta = \delta^1(c) = \delta_R^1(c)$, we have $\mathsf{M}^1(\delta) = \mathsf{M}^1(c)$, and the cover $\mathsf{c}$ consisting of all the finite subsets of $X$ belongs to $\mathsf{M}^1(\delta)\backslash\mathsf{M}_R^1(\delta)$. ($\mathsf{c}$ is a $\delta$-cover, so $\mathsf{c} \in \mathsf{M}^1(\delta)$ by Theorem 5.5. $\mathsf{c} \notin \mathsf{M}_R^1$, because $z \notin \bigcup\operatorname{int}\mathsf{c}$). ◇

**5.12** Similarly to 5.7 and 5.8, it is possible to deduce from each other Theorem 1.5 and the part of Theorem 3.2 concerning coarsest extensions. (Make use of Lemma 5.3 d).) In addition to the formulas given in 5.7 and 5.8, we have (for a family of merotopies, repsectively proximities, in a weakly separated closure space, with Cauchy, respectively compressed, trace filters):

(1) $$\mathsf{M}^1_R(c,\mathsf{M}_i)=\mathsf{M}^1_R(\delta^1_R(c,\delta(\mathsf{M}_i)),\mathsf{M}_i);$$

(2) $$\delta^1_R(c,\delta_i)=\delta(M^1_R(c,\mathsf{M}^0(\delta_i))).$$

## C. LODATO MEROTOPIES IN A PROXIMITY SPACE

**5.13** If a family of merotopies in a proximity sapce has a Lodato extension then the proximity and the merotopies are Lodato, the trace filters are Cauchy, and 3.6 (1) holds, since an extension in $(X,\delta)$ is necessarily an extension in $(X,c)$. These conditions are not sufficient, not even for a single open subset:

**Example.** Take $X, X_1$ and $\mathsf{M}_1$ from Example 3.8, and let $\delta$ be the Euclidean proximity on $X$. Now $\mathsf{M}_1$ and $\delta$ are Lodato, $\mathsf{M}_1$ is compatible with $\delta|X_1$, the trace filters are Cauchy, 3.6 (1) is evident (cf. Corollary 3.7), both $\mathcal{U}(\mathsf{M}_1)$ and $\Gamma(\mathsf{M}_1)$ have Lodato extensions, but $\mathsf{M}_1$ does not have one:

Assume indirectly that $\mathsf{N}$ is a Lodato extension. Then $\mathsf{c}_1(1)^0\in\mathsf{N}$, and so $\mathsf{d}=\operatorname{int}\mathsf{c}_1(1)^0\in\mathsf{N}$; now $\mathsf{d}|X_1^r$ consists of singletons, implying that $\delta|X_1^r$ is discrete, a contradiction. ◊

**5.14 Definition.** For a family of Lodato merotopies in a Lodato proximity space with Cauchy trace filters, let $\{\operatorname{int}\mathsf{c}:\mathsf{c}\in\mathsf{B}\}$ be a subbase for $\mathsf{M}^0_L$ (with $\mathsf{B}$ from Definition 5.3). ◊

In other words, $\{\operatorname{int}\mathsf{c}:\mathsf{c}\in\mathsf{M}^0\}$ is a base for $\mathsf{M}^0_L$. ($\operatorname{int}\mathsf{c}$ is a cover by Theorem 5.9.) $\operatorname{int}\mathsf{c}_{A,B}=\mathsf{c}_{c(A),c(B)}$, so the following covers form a subbase $\mathsf{B}_L$ for $\mathsf{M}^0_L$:

(1) $\mathsf{c}_{A,B}$ $\quad$ ($A\bar{\delta}B$, $A$ and $B$ are $c$-closed);

(2) $\operatorname{int}\mathsf{c}^0_i$ $\quad$ ($i\in I$, $\mathsf{c}_i\in\mathsf{M}_i$, $\mathsf{c}_i$ is $c_i$-open).

The covers in this subbase are clearly open in $c$. $\mathsf{M}^0_L$ is finer than the compatible merotopy $\mathsf{M}^0$. On the other hand, the $c$-openess of the covers implies that $c(\mathsf{M}^0_L)$ is coarser than $c$; therefore:

**Lemma.** *Under the assumptions of the definition, $\mathsf{M}^0_L$ is a Lodato merotopy compatible with $c$.* ◊

$\mathsf{M}_L^0$ is not necessarily compatible with $\delta$, see Example 5.13. We shall also see that $\mathsf{M}_L^0|X_i$ can be different from $\mathsf{M}_i$ (see Examples 5.19).

**5.15 Lemma.** *If $\delta$ is a Lodato proximity then $\mathsf{M}^0(\delta) = \mathsf{M}_L^0(\delta)$ is the coarsest Lodato merotopy compatible with $\delta$.*

**Proof.** $\mathsf{M}^0 \subset \mathsf{M}_L^0$ always holds, while the converse follows for $I = \emptyset$ from $\mathsf{B}_L \subset \mathsf{B}$. Now Lemma 5.14 and Theorem 5.4 can be applied. ◇

**5.16 Lemma.** *Under the assumptions of Definition 5.14, $\mathsf{M}_L^0$ is the coarsest one among those Lodato merotopies $\mathsf{M}$ compatible with $c$ that induce a proximity finer than $\delta$, and for which $\mathsf{M}|X_i$ is finer than $\mathsf{M}_i$ $(i \in I)$.*

**Proof.** $\delta(\mathsf{M}_L^0)$ is finer than $\delta$, because $\mathsf{M}_L^0$ is finer than $\mathsf{M}_L^0(\delta)$, and the latter is compatible by Lemma 5.15. $\mathsf{M}_L^0|X_i \supset \mathsf{M}_i$, because if $\mathsf{c}_i \in \mathsf{M}_i$ is $c_i$-open then $\mathsf{c}_i = (\operatorname{int} \mathsf{c}_i^0)|X_i$. $\mathsf{M}_L^0$ is Lodato and $c(\mathsf{M}_L^0) = c$ (Lemma 5.14).

Let $\mathsf{M}$ be a merotopy satisfying the conditions of the lemma; we have to show that $\mathsf{B}_L \subset \mathsf{M}$.

If $A\bar{\delta}B$ then $A\overline{\delta(\mathsf{M})}B$, so $\mathsf{c}_{A,B} \in \mathsf{M}^0(\delta(\mathsf{M})) \subset \mathsf{M}$ by Theorem 5.4. $\mathsf{M}|X_i \supset \mathsf{M}_i$ implies that for any $c_i$-open cover $\mathsf{c}_i \in \mathsf{M}_i$ there is a $\mathsf{c} \in \mathsf{M}$ with $\mathsf{c}|X_i = \mathsf{c}_i$; $\operatorname{int} \mathsf{c} \in \mathsf{M}$ (as $\mathsf{M}$ is Lodato, and it is compatible with $c$); now $\operatorname{int} \mathsf{c}$ refines $\operatorname{int} \mathsf{c}_i^0$, thus $\operatorname{int} \mathsf{c}_i^0 \in \mathsf{M}$, too. ◇

It has to be assumed in the lemma that $\mathsf{M}$ is compatible with $c$:

**Example.** On $X = \mathbb{N}^2$, let $A\bar{\delta}B$ iff their projections on the first coordinate are disjoint. Take the discrete merotopy $\mathsf{M}_0$ on $X_0 = \mathbb{N} \times \times \{1\}$, and let $\mathsf{M}$ be the merotopy for which $\mathsf{M}^0(\delta') \cup \{\mathsf{c}_0^0\}$ constitutes a subbase, where $\delta'$ is the discrete proximity on $X$, and $\mathsf{c}_0$ consists of the singletons in $X_0$. Now $\mathsf{M}$ is not compatible with $c$, but the other conditions of the theorem are satisfied. $\mathsf{M}_L^0$ is not coarser than $\mathsf{M}$, because $\mathsf{M}|X_0^r$ is contigual, while $(\operatorname{int} \mathsf{c}_0^0)|X_0^r \in \mathsf{M}_L^0|X_0^r$ cannot be refined by a finite cover. ◇

**5.17 Lemma.** *A family of merotopies in a proximity space has a Lodato extension iff*

(i) *the proximity and the merotopies are Lodato;*

(ii) $(\bigcap_{i \in F})\operatorname{int} \mathsf{c}_i^0$ *is a $\delta$-cover whenever $\emptyset \neq F \subset I$ is finite, and* $\mathsf{c}_i \in \mathsf{M}_i$ $(i \in F)$;

(iii) $(\operatorname{int} \mathsf{c}_i^0)|X_j \in \mathsf{M}_j$ $(i, j \in I, \mathsf{c}_i \in \mathsf{M}_i)$.

*If these conditions are satisfied then* $\mathsf{M}_L^0$ *is the coarsest Lodato extension.*

**Remarks.** a) It is not necessary to assume that the trace filters are Cauchy, since this follows from (ii). (Recall that the trace filters are Cauchy iff each $\operatorname{int} \mathsf{c}_i^0$ is a cover.)

b) It is enough to know (ii) and (iii) for elements of bases for $\mathsf{M}_i$, e.g. for open covers.

c) The cover in (ii) can also be written as $\operatorname{int}(\bigcap_{i\in F})\mathsf{c}_i^0$.

**Proof.** 1° *Necessity.* It was already mentioned in 5.13 that (i) and (iii) are necessary. If there is a Lodato extension then $\mathsf{M}_L^0$ is an extension by Lemma 5.16. The covers in (ii) belong to $\mathsf{M}_L^0$, so Lemma 5.1 implies that they are $\delta$-covers.

2° *Sufficiency.* The assumptions of Definition 5.14 are fullfilled, see Remark a). $\delta(\mathsf{M}_L^0) \subset \delta$ and $\mathsf{M}_L^0|X_i \supset \mathsf{M}_i$ by Lemma 5.16. Conversely, $\delta(M_L^0) \supset \delta$ follows from Lemma 5.1, since the elements of the base generated by $\mathsf{B}_L$ are $\delta$-covers by (ii) and Lemmas 5.3 a) and 5.2; $\mathsf{M}_L^0|X_i \subset \mathsf{M}_i$ follows from (iii) and Lemma 5.2 b) and c). Thus $\mathsf{M}_L^0$ is an extension, Lodato by Lemma 5.14. ◊

**Corollary.** *A single Lodato merotopy* $\mathsf{M}_0$ *in a Lodato proximity space has a Lodato extension iff* $\operatorname{int} \mathsf{c}_0^0$ *is a* $\delta$*-cover for each (*$\mathsf{c}_0$*-open)* $\mathsf{c}_0 \in \mathsf{M}_0$*; if so then* $\mathsf{M}_L^0$ *is the coarsest extension.* ◊

It can occur that a single merotopy in a proximity space has a Lodato extension, but $\mathsf{M}_L^0 \neq \mathsf{M}^0$ (we have seen in Lemma 5.15 that this is impossible for $I = \emptyset$):

**Example.** Let $X = \mathbb{R} \times [0, \to[$, with the Euclidean proximity $\delta$, $X_0 = \mathbb{R} \times ]0, \to[$, $\mathsf{M}_0$ the Euclidean merotopy on $X_0$. Now $\mathsf{M}_0$ has a Lodato extension (the Euclidean merotopy on $X$, which is in fact equal to $\mathsf{M}_L^0$), but $\mathsf{M}_L^0 \neq \mathsf{M}^0$, since $\mathsf{M}^0|X_0^r$ is contigual, while $\mathsf{M}_L^0|X_0^r$ is not contigual. ◊

**5.18** $\operatorname{int} \mathsf{c}_0^0$ clearly satisfies the condition in Definition 5.1 for $A, B \subset \subset X_0$, so, in view of Axiom P5, it is enough to assume this condition in Corollary 5.17 for $A \subset X_0^r$ and for $B$ satisfying $B \subset X_0$ or $B \subset X_0^r$. Thus the assumption in Corollary 5.17 splits into two parts:

(a) if $A, B \subset X_0^r$, $A\delta B$ and $\mathsf{c}_0 \in \mathsf{M}_0$ (is open) then there are $C_0 \in \mathsf{c}_0$, $x \in A$ and $y \in B$ such that $C_0 \in \mathsf{s}_0(x) \cap \mathsf{s}_0(y)$;

(b) if $A \subset X_0^r$, $B \subset X_0$, $A\delta B$ and $\mathsf{c}_0 \in \mathsf{M}_0$ (is open) then there are $C_0 \in \mathsf{c}_0$ and $x \in A$ such that $C_0 \in \mathsf{s}_0(x)$ and $C_0 \cap B \neq \emptyset$.

Either of these conditions implies that the trace filters are Cauchy. (For $x \in c(X_0)\setminus X_0$, take $A = \{x\}$ and either $B = \{x\}$ or $B = X_0$.) The next examples show that neither is sufficient in itself for the existence of a Lodato extension.

**Examples.** a) Modify Example 5.13, replacing each $\mathsf{c}_1(\varepsilon)$ by

$$\{C_1 \cup D_1 : C_1,\ D_1 \in \mathsf{c}_1(\varepsilon),\ C_1 \cap D_1 \neq \emptyset\}.$$

Now (b) holds, but there is no Lodato extension, for the same reason as in 5.13.

b) Let $X = \mathbb{N} \times [0,\rightarrow[$, $X_0 = \mathbb{N}\times]0,\rightarrow[$. Take the Euclidean proximity $\delta$ on $X$, and let the following covers ($n \in \mathbb{N}$) constitute a base for $\mathsf{M}_0$ on $X_0$:

$$\big\{\{k\}\times]y, y+\tfrac{1}{n}[: k \in \mathbb{N}, y > 0\big\} \cup \big\{\{k\}\times]0, \tfrac{1}{\max\{k,n\}}[: k \in \mathbb{N}\big\}.$$

Now (a) holds, $\mathcal{U}(\mathsf{M}_0)$ and $\Gamma(\mathsf{M}_0)$ have Lodato extensions in $(X,\delta)$ (observe that $\mathcal{U}(\mathsf{M}_0) = \mathcal{U}(\mathsf{N}_0)$ and $\Gamma(\mathsf{M}_0) = \Gamma(\mathsf{N}_0)$, where $\mathsf{N}_0$ is the Euclidean merotopy on $X_0$), but $\mathsf{M}_0$ does not have one, since (b) fails for $A = X_0^r$ and $B = \{(k, 1/k) : k \in \mathbb{N}\}$. ◊

**5.19** Condition (iii) is not superfluous in Lemma 5.17:

**Examples.** a) Let $X, X_0, X_1, \mathsf{M}_0, \mathsf{M}_1$ be as in Example 3.8, with the following modification: replace $\mathsf{c}_1(\varepsilon)$ by

$$\mathsf{d}_1(\varepsilon) = \mathsf{c}_1(\varepsilon) \cup \big\{(\{1/m, 1/n\}\times]0,\varepsilon[) \cap X_1 : m, n \in \mathbb{N}, m, n > 1/\varepsilon\big\}.$$

Let $\delta$ be the Euclidean proximity on $X$. 5.17 (i) is clearly staisfied.

$\operatorname{int} \mathsf{d}_1(\varepsilon)^0$ is a $\delta$-cover (the modification was needed, because otherwise neither 5.18 (a) nor 5.18 (b) would hold). For $\mathsf{c}_0 \in \mathsf{M}_0$, $\operatorname{int} \mathsf{c}_0^0$ is evidently a $\delta$-cover, since $X_0$ is closed. $\mathsf{M}_0$ is contigual, so $\operatorname{int} \mathsf{c}_0^0$ is finite for $\mathsf{c}_0$ taken from a base. Hence (ii) holds by Lemma 5.2. The induced semi-uniformities as well as the induced contiguities have an extension (similarly to 3.8, the Euclidean uniformity, respectively the Euclidean contiguity). But $\mathsf{M}_0$ and $\mathsf{M}_1$ do not have a Lodato extension, not even in $(X, c)$, since (iii) is not satisfied for $i = 1$, $j = 0$, $\mathsf{c}_i = \mathsf{d}_1(1)$.

b) There is a much simpler example if we do not insist that the induced semi-uniformities should have a Lodato extension (essentially the same as Example 2.10):

Let $X, X_0, \delta, \mathsf{M}_0$ be as in Example 5.17, $X_1 = X_0^r$, $\Gamma_1$ the Euclidean contiguity on $X_1$, $\mathsf{M}_1 = \mathsf{M}^0(\Gamma_1)$ (cf. 4.1). ◊

**5.20** Condition (ii) of Lemma 5.17 cannot be replaced by the weaker assumption that each $\mathrm{int}\,\mathsf{c}_i^0$ is a $\delta$-cover:

**Example.** Let $T = \{-1/n, 1/n : n \in \mathbb{N}\}$, $X = T \times \mathbb{R}$, $X_0 = T\times]\leftarrow, 0[$, $X_1 = T\times]0, \rightarrow[$. Let $\delta$ be the Euclidean proximity on $X$, and $\{\mathsf{c}_i(\varepsilon) : \varepsilon > 0\}$ a base for $\mathsf{M}_i$ on $X_i$, where

$$\begin{aligned} \mathsf{c}_1(\varepsilon) = \big\{(]p, p+\varepsilon[\times]q, q+\varepsilon[) \cap X_1 : (p \in \mathbb{R}, q > 0) \text{ or} \\ (0 \notin ]p, p+\varepsilon[, q = 0)\big\} \cup \big\{\{-1/k, 1/n\}\times]0, \varepsilon[ : k > n > 1/\varepsilon\big\}, \\ \mathsf{c}_0(\varepsilon) = \big\{(]p, p+\varepsilon[\times]q-\varepsilon, q[) \cap X_0 : (p \in \mathbb{R}, q < 0) \text{ or} \\ (0 \notin ]p, p+\varepsilon[, q = 0)\big\} \cup \big\{\{-1/k, 1/n\}\times]-\varepsilon, 0[ : n > k > 1/\varepsilon\big\}, \end{aligned}$$

(i) and (iii) are fullfilled, the latter because, for $i \neq j$, $\mathrm{int}\,\mathsf{c}_i^0|X_j = \{X_j\}$. The weaker form of (ii) holds, but not (ii) itself, since $\mathrm{int}\,\mathsf{c}_0(1)^0(\cap)$ $(\cap)\mathrm{int}\,\mathsf{c}_1(1)^0$ is not a $\delta$-cover: consider $A = \{1/n : n \in \mathbb{N}\} \times \{0\}$ and $B = \{-1/n : n \in \mathbb{N}\} \times \{0\}$. $\diamondsuit$

**5.21** In the extension problems we have discussed up to now, a family of structures could be extended iff each subfamily of cardinality $\leq 2$ had an extension. We do not know whether this holds for Lodato extensions of merotopies in a proximity space.

**5.22 Theorem.** *A family of Lodato merotopies given on closed subsets in a Lodato proximity space has Lodato extensions;* $\mathsf{M}^0 = \mathsf{M}_L^0$ *is the coarsest one.*

**Proof.** $\mathsf{M}^0$ is the coarsest extension by Theorem 5.4. $\mathsf{M}^0$ is Lodato, since $\mathsf{c}_i^0$ is refined by $(\mathrm{int}_i\,\mathsf{c}_i)^0 \in \mathsf{M}^0$, which is an open cover, and $\mathsf{c}_{A,B}$ is refined by the open cover $\mathsf{c}_{c(A),c(B)} \in \mathsf{M}^0$, thus $\mathsf{M}^0$ has a subbase consisting of open covers. $\mathsf{M}^0 = \mathsf{M}_L^0$ is also clear from this reasoning. $\diamondsuit$

If the subsets are not closed then it is possible that there exist Lodato extensions, but $\mathsf{M}_L^0$ (by Lemma 5.16, the coarsest one) is strictly finer than $\mathsf{M}^0$:

**Example.** Take $S = \{1/n : n \in \mathbb{N}\}$, $X = S\times(\{0\} \cup S)$, $X_1 = S^2$. Let $\delta$ be the Euclidean proximity on $X$, and $\{\mathsf{f}_1(k) : k \in \mathbb{N}\}$ a subbase for $\mathsf{M}_1$ on $X_1$, with $\mathsf{f}_1(k)$ from Example 4.5. Now $\mathsf{M}_L^0$ is a Lodato extension, and $\mathrm{int}\,\mathsf{f}_1(1)^0 \in \mathsf{M}_L^0 \setminus \mathsf{M}_0$. $\diamondsuit$

**5.23 Lemma.** *If a family of merotopies in a proximity space has a Lodato extension, and the open $\delta$-covers* $\mathsf{c}$ *for which* $\mathsf{c}|X_i \in \mathsf{M}_i$ $(i \in I)$ *form a base for a merotopy* $\mathsf{M}_L^1$ *then* $\mathsf{M}_L^1$ *is the finest Lodato extension.* $\diamondsuit$

It follows from this lemma and Theorem 5.10 that if a family of merotopies has a Lodato extension as well as a finest Riesz extension then it has a finest Lodato extension, too; the converse is not true:
**Example.** With $X, P, Q$ from Example 5.2, let $c$ denote the topological sum of the cofinite topologies on $P$ and $Q$. Define $A\delta B$ iff either $c(A)\cap$ $\cap c(B) \neq \emptyset$, or $A \cap P$ and $B \cap Q$ are infinite, or $A \cap Q$ and $B \cap P$ are infinite. $\delta$ is a Lodato proximity compatible with $c$. An open cover $\mathsf{c}$ is a $\delta$-cover iff there is a $C \in \mathsf{c}$ with $C^r$ finite; if the open covers $\mathsf{c}$ and $\mathsf{d}$ have this property then so has $\mathsf{c}(\cap)\mathsf{d}$, thus the open $\delta$-covers constitute a base for a merotopy, which is, according to the lemma, the finest compatible Lodato merotopy.

There is, however, no finest compatible Riesz merotopy, because, by Theorem 5.10, such a merotopy would contain $\mathsf{c}$ and $\mathsf{d}$ from Example 5.2; but $\mathsf{c}(\cap)\mathsf{d}$ is clearly not a $\delta$-cover, a contradiction. $\diamondsuit$
**Problem.** Assume that there exists a finest Lodato extension; is it necessarily of the form given in Lemma? (The answer is yes if each $X_i$ is closed: repeat the reasoning from the second paragraph of the proof of Theorem 5.5, considering only $c$-open, respectively $c_i$-open covers; if $\mathsf{c}_i$ is $c_i$-open and $X_i$ is closed then $\mathsf{c}_i^0$ is $c$-open.)

**5.24** We need a measurable cardinal in the construction of a proximity space in which the finest compatible Lodato and Riesz merotopies exist but differ (compare with the very simple examples in 5.11):
**Example.** Let $Y$ be the set of the rationals, $Z$ a set of measurable cardinality, $Y \cap Z = \emptyset$, $X = Y \cup Z$, $\mathsf{u}$ a free ultrafilter on $Z$ such that $\cap\mathsf{v} \in \mathsf{u}$ whenever $\mathsf{v} \subset \mathsf{u}$ is countable (see e.g. [4] 12.2). Let $c$ denote the sum of the Euclidean topology on $Y$ and the discrete one on $Z$. Define $A\delta B$ iff either $c(A) \cap c(B) \neq \emptyset$, or $A \cap Y$ is infinite and $B \cap Z \in \mathsf{u}$, or $B \cap Y$ is infinite and $A \cap Z \in \mathsf{u}$. $\delta$ is a Lodato proximity compatible with $c$. Let $\mathsf{c}$ and $\mathsf{d}$ be $\delta$-covers for which $\operatorname{int}\mathsf{c}$ and $\operatorname{int}\mathsf{d}$ are covers. Evidently, $\operatorname{int}(\mathsf{c}(\cap)\mathsf{d})$ is also a cover. We are going to show that $\mathsf{c}(\cap)\mathsf{d}$ is a $\delta$-cover; then Theorem 5.10 yields that there exists the finest compatible Riesz merotopy $\mathsf{M}_R^1(\delta)$, implying the existence of the finest compatible Lodato merotopy $\mathsf{M}_L^1(\delta)$.

Given near sets $A$ and $B$, we need $C \in \mathsf{c}$ and $D \in \mathsf{d}$ such that

$$A \cap C \cap D \neq \emptyset \neq B \cap C \cap D. \tag{1}$$

If there is a point $x \in c(A) \cap c(B)$ then, as $\operatorname{int}\mathsf{c}$ and $\operatorname{int}\mathsf{d}$ are covers, $C$ and $D$ can be chosen such that $x \in \operatorname{int} C \cap \operatorname{int} D$, and then (1) clearly holds. So we may assume without loss of generality that $A \subset Y$ and $B \subset Z$, $A$ is infinite and $B \in \mathsf{u}$.

We shall define by recursion sets $A_n \subset A_1 = A$, $B_n \subset B_1 = B$ satisfying $A_n \delta B_n$, and points $x_n \in A$ ($n \in \mathbb{N}$). If $A_n$ and $B_n$ are defined then consider the sets $B_n \setminus \mathrm{St}(x, \mathsf{c})$ for $x \in A_n$. If all these sets belonged to $\mathsf{u}$ then we would have $E = B_n \setminus \mathrm{St}(A_n, \mathsf{c}) \in \mathsf{u}$; now $A_n \delta E$, contradicting the assumption that $\mathsf{c}$ is a $\delta$-cover. Hence there is an $x_n \in A_n$ such that $Z \cap \mathrm{St}(x_n, c) \in \mathsf{u}$; define now $A_{n+1} = A_n \setminus \{x_n\}$ and $B_{n+1} = B_n \cap \mathrm{St}(x_n, \mathsf{c})$; clearly, $A_{n+1} \delta B_{n+1}$, and the points $x_n$ are different. Take $H = \{x_n : n \in \mathbb{N}\}$ and $K = \bigcap_{n \in \mathbb{N}} B_n$; then $H \delta K$, and

$$\mathrm{St}(y, \mathsf{c}) \supset K \qquad (y \in H). \tag{2}$$

$\mathsf{d}$ being a $\delta$-cover, there is a $D \in \mathsf{d}$ such that $D \cap H \neq \emptyset \neq D \cap K$. Taking points $y \in D \cap H$ and $z \in D \cap K$, (2) implies that $y, z \in C$ for some $C \in \mathsf{c}$, i.e. (1) holds indeed.

Consider the cover

$$\mathsf{e} = \{Y\} \cup \big\{\{y\} \cup Z : y \in Y\big\}.$$

$\mathrm{int}\,\mathsf{e} = \{Y, Z\}$ is a cover, and $\mathsf{e}$ is a $\delta$-cover, thus $\mathsf{e} \in \mathsf{M}^1_R(\delta)$ by Theorem 5.10. But $\mathsf{e} \notin \mathsf{M}^1_L(\delta)$, since $\mathrm{int}\,\mathsf{e}$ is not a $\delta$-cover. Hence $\mathsf{M}^1_R(\delta) \neq$ $\neq \mathsf{M}^1_L(\delta)$. ◊

**Problem.** Is there a similar example in ZFC, or at least in a consistent model of ZFC? (Perhaps there exists such an example only with $I \neq \emptyset$.)

**5.25** It follows easily from the definition that under the conditions of Definition 5.14,

$$\mathsf{M}^0_L = \sup_{i \in I} \mathsf{M}^0_L(\delta, \{\mathsf{M}_i\}) \tag{1}$$

holds for $I \neq \emptyset$. (1) cannot be deduced from 2.2 a) 1° in such generality, since it holds only for $p = q = 1$ that $\mathsf{M}^0_L$ is always a $pq$-overextension (see the last paragraph in 5.14), but it is not the coarsest one (Example 5.16). We can, however, generalize 2.2 a) 1° to meet the present situation (with $p = q = 1$; cf. Lemma 5.16): let us require in the definition of a $pq$-overextension that $d$ should satisfy a property inherited by suprema of non-empty collections. (The **C**-structure on $X$ is allowed to figure in the property.)

**5.26** Statements similar to those in 5.7 and 5.8 hold for Lodato extensions, too. It should be mentioned that extending a family of merotopies in a closure space in two steps is now even more problematic (because Lodato merotopies behave badly in a proximity space), e.g. Corollary 3.8 can be obtained this way only for closed subsets, and not for open ones.

## 6. Extending a family of contiguities in a proximity space

### A. WITHOUT SEPARATION AXIOMS

**6.1** A family of contiguities in a proximity space always has extensions; this will be deduced from the corresponding result for merotopies, using the method of § 4. We shall utilize the facts mentioned in the second paragraph of 4.1. Only the coarsest extension can be obtained this way, although there exists a finest one, too; its existence can be proved easily: take the supremum $\Gamma$ of all the extensions (i.e. their union is a subbase for $\Gamma$); now $\Gamma$ is compatible by the lemma below, and $\Gamma|X_i = \Gamma_i$ is evident. This proof is, however, superfluous, since we shall construct the finest extension.

**Lemma.** *For a contiguity $\Gamma$ on $X$, $\delta(\Gamma)$ is coarser than $\delta$ iff every $\mathsf{f} \in \Gamma$ is a $\delta$-cover iff $\Gamma$ has a subbase consisting of $\delta$-covers.*

**Proof.** The statement on subbases follows from Lemma 5.2. ◊

**6.2 Definition.** For a family of contiguities in a proximity space,

a) Let $\Gamma^0$ be the contiguity for which the following covers form a subbase: $\mathsf{f}_i^0$ ($i \in I$, $\mathsf{f}_i \in \Gamma_i$) and $\mathsf{c}_{A,B}$ ($A\bar{\delta}B$).

b) Let $\Gamma^1$ consist of those finite $\delta$-covers $\mathsf{f}$ for which $\mathsf{f}|X_i \in \Gamma_i$ ($i \in I$). ◊

Clearly, $\Gamma^0 = \Gamma(\mathsf{M}^0(\delta, \mathsf{M}^0(\Gamma_i)))$.

**Theorem.** *A family of contiguities in a proximity space always has extensions. $\Gamma^0$ is the coarsest, and $\Gamma^1$ the finest extension.*

**Remark.** A direct proof not making use of Theorem 5.4 would be much simpler than the proof of that theorem, since, the covers being finite, the argument in 5.4 2° can be replaced by applying Lemma 5.2 (or 6.1).

**Proof.** $\Gamma^0$ is an extension by Theorem 5.4. If $\Gamma$ is another extension then $\mathsf{M}^0(\Gamma)$ is an extension of the merotopies $\mathsf{M}^0(\Gamma_i)$, hence $\mathsf{M}^0(\delta, \mathsf{M}^0(\Gamma_i)) \subset \mathsf{M}^0(\Gamma)$, thus $\Gamma^0 \subset \Gamma$. If $\mathfrak{f} \in \Gamma$ then it satisfies the conditions in Part b) of the definition, so $\mathfrak{f} \in \Gamma^1$, and therefore $\Gamma \subset \Gamma^1$; in particular, $\Gamma^0 \subset \Gamma^1$, implying that $c(\Gamma^1)$ is finer than $c$ and $\Gamma_i \subset \Gamma^1|X_i$. Conversely, $c(\Gamma^1)$ is coarser than $c$ by Lemma 6.1, and $\Gamma^1|X_i \subset \Gamma_i$ is evident from the definition. Thus $\Gamma^1$ is indeed the finest extension. ◇

**6.3** $\Gamma^0$ and $\Gamma^1$ are different in general: let $|X| = 3$, $I = \emptyset$ and $\delta$ the indiscrete proximity on $X$. $\Gamma^0$ and $\Gamma^1$ can, in fact, coincide only under very strong assumptions: $\Gamma^0(\delta) = \Gamma^1(\delta)$ iff each $\delta$-compressed filter is the intersection of at most two ultrafilters; this will be proved in [3], along with the following results: all the $\delta$-covers of cardinality $\leq 3$ form a subbase for $\Gamma^1(\delta)$; if proximities $\delta[i]$ ($i \in I \neq \emptyset$) are given on the same set then

$$\sup_{i \in I} \Gamma^1(\delta[i]) = \Gamma^1(\sup_{i \in I} \delta[i]).$$

**6.4** The analogue for contiguities of 5.6 (1) and a similar formula for $\Gamma^1$ follow easily from 2.2 a).

Statements corresponding to 5.7 and 5.8 are also valid; things are simplified by the existence of a finest extension. Only one point is worth going into: the formulas

$$\delta^k(c, \delta_i) = \delta(\Gamma^k(c, \Gamma^0(\delta_i))) \qquad (k = 0, 1) \tag{1}$$

remain valid if we substitute $\Gamma^1(\delta_i)$ for $\Gamma^0(\delta_i)$. The formulas make sense, because the contiguities $\Gamma^1(\delta_i)$ are accordant. It follows from (1) that $\delta(\Gamma^1(c, \Gamma^1(\delta_i)))$ is finer than $\delta^1(c, \delta_i)$, so they are the same, as the latter is the finest extension, and the former is an extension, too. Concerning the case $k = 0$, observe that $\Gamma^0(c, \Gamma^1(\delta_i)) \subset \Gamma^1(\delta^0(c, \delta_i))$, since (see Definition 4.1 a)) $\mathfrak{c}_{x,B}$ belongs to any contiguity compatible with $c$, while if $\mathfrak{f}_j$ is a finite $\delta_j$-cover then $\mathfrak{f}_j^0$ is a finite $\delta^0(c, \delta_i)$-cover; hence

$$\delta(\Gamma^0(c, \Gamma^1(\delta_i))) \supset \delta^0(c, \delta_i) = \delta(\Gamma^0(c, \Gamma^0(\delta_i))) \supset \delta(\Gamma^0(c, \Gamma^1(\delta_i))).$$

## B. RIESZ CONTIGUITIES IN A PROXIMITY SPACE

**6.5 Definition.** For a family of contiguities in a proximity space, let

$$\Gamma^1_R = \{\mathsf{f} \in \Gamma^1 : \operatorname{int} \mathsf{f} \text{ is a cover of } X\}. \ \diamondsuit$$

(The same definition was used in a closure space, with a different meaning of $\Gamma^1$, of course, see 4.2.)

**Theorem.** *A family of contiguities in a proximity space has a Riesz extension iff the proximity is Riesz and the trace filters are Cauchy; if so then $\Gamma^0$ is the coarsest and $\Gamma^1_R$ the finest Riesz extension.*

**Proof.** In view of Theorem 5.9, it is enough to show that $\Gamma^1_R$ is the finest Riesz extension. If $\Gamma$ is a Riesz extension then $\Gamma \subset \Gamma^1$ by Theorem 6.2, so $\Gamma \subset \Gamma^1_R$ follows from the definition. In particular, $\Gamma^0 \subset \Gamma^1_R$; on the other hand, $\Gamma^1_R \subset \Gamma^1$ is evident, thus $\Gamma^1_R$ is an extension by Theorem 6.2. $\Gamma^1_R$ is clearly Riesz, and we have already seen that it is finer than any other Riesz extension. $\diamondsuit$

$\Gamma^0$, $\Gamma^1_R$ and $\Gamma^1$ can be different:

**Example.** Let $\delta$ be the Euclidean proximity on $X = \mathrm{R}\setminus\{0\}$. Denote by $Q$ and $D$ the set of the rationals, respectively dyadic rationals, in $X$. Now

$$\mathsf{f} = \{Q, D^r, D \cup Q^r\} \in \Gamma^1(\delta)\setminus\Gamma^1_R(\delta),$$
$$\mathsf{f}' = \{] \leftarrow, 0[, ]0, \rightarrow [\} \cup \mathsf{f} \in \Gamma^1_R(\delta)\setminus\Gamma^0(\delta). \ \diamondsuit$$

**6.6** It follows from 2.2 a) 3° and 4° that, under the assumptions of Theorem 6.5,

$$\text{(1)} \qquad \Gamma^1_R = \inf_{i\in I} \Gamma^1_R(\delta, \{\Gamma_i\}) = \inf\{\Gamma^1_R(\delta), \inf_{i\in I} \Gamma^{11}[i]\},$$

where $\Gamma^{11}[i]$ is the finest contiguity (= the finest Riesz contiguity) $\Gamma$ on $X$ for which $\Gamma|X_i = \Gamma_i$, i.e. $\Gamma^{11}[i]$ consists of all those finite covers $\mathsf{f}$ of $X$ for which $\mathsf{f}|X_i \in \Gamma_i$. ($\Gamma^{11}[i]$ is Riesz because $\operatorname{int}'A = (A\setminus X_i) \cup \operatorname{int}_i(A \cap X_i)$, where $\operatorname{int}'$ is to be understood in $c(\Gamma^{11}[i])$.) (1) is in fact obtained with inf taken in the category of Riesz contiguities, but this coincides with inf in the category of contiguities, assuming that there exists a coarsest one among the closures induced by the contiguities considered. (And observe that $\delta(\Gamma^{11}[i]) \subset \delta$, implying that $c(\Gamma^{11}[i])$ is finer than $c = c(\Gamma^1_R(\delta))$.)

**6.7** The finest Riesz extension of a family of contiguities in a closure space can be obtained in two steps, cf. 5.12 (1) (but now the existence of a finest extension can in fact be *proved* in two steps):

$$\text{(1)} \qquad \Gamma^1_R(c, \Gamma_i) = \Gamma^1_R(\delta^1_R(c, \delta(\Gamma_i)), \Gamma_i).$$

Conversely, if we have a family of proximities in a weakly separated closure space such that the trace filters are compressed then it follows from 5.12 (2) that

$$\delta^1_R(c, \delta_i) = \delta(\Gamma^1_R(c, \Gamma^0(\delta_i))). \tag{2}$$

If we try to replace here $\Gamma^0(\delta_i)$ by $\Gamma^1_R(\delta_i)$ (cf. 6.4) then the trace filters are not necessarily Cauchy, thus $\Gamma^1_R(c, \Gamma^1_R(\delta_i))$ is not a Riesz extension (in fact, not an extension at all, see the example below); all the same, (2) remains valid even with $\Gamma^1(\delta_i)$, since $\Gamma^1_R(c, \Gamma^0(\delta_i))$ and $\Gamma^1_R(c, \Gamma^1(\delta_i))$ induce the same proximity (using Definition 4.2, check that if $\mathsf{f} \in \Gamma^1_R(c, \Gamma^1(\delta_i))$ and $|\mathsf{f}| = 2$ then $\mathsf{f} \in \Gamma^1_R(c, \Gamma^0(\delta_i))$); and $\Gamma^1_R(c, \Gamma^1_R(\delta_i)) = \Gamma^1_R(c, \Gamma^1(\delta_i))$.

**Example.** Let $X = \mathbb{N}$, $X_0 = \{1\}^r$, $S \in \mathsf{v}(1)$ iff $1 \in S$ and $S^r$ is finite, and let the other points be isolated in $c$. For disjoint $A, B \subset X_0$, define $A \delta_0 B$ iff $A$ and $B$ are infinite. Take disjoint infinite sets $A, B, C \subset \subset X_0$. Now $\mathsf{f}_0 = \{X_0 \setminus A, X_0 \setminus B, X_0 \setminus C\} \in \Gamma^1_R(\delta_0)$, so the $\delta_0$-compressed filter $\mathsf{s}_0(1)$ is not $\Gamma^1_R(\delta_0)$-Cauchy, because $\mathsf{s}_0(1) \cap \mathsf{f}_0 = \emptyset$. Moreover, $\Gamma^1_R(c, \Gamma^1_R(\delta_0))$ is not an extension, since if $\mathsf{f}$ belongs to it then $1 \in \bigcup \mathrm{int}\, \mathsf{f}$ implies that $\mathsf{f}|X_0$ contains a cofinite set, i.e. $\mathsf{f}|X_0 \neq \mathsf{f}_0$. $\diamondsuit$

## C. LODATO CONTIGUITIES IN A PROXIMITY SPACE

**6.8 Definition.** For a family of contiguities in a proximity space,

a) Let $\Gamma^1_L = \{\mathsf{f} \in \Gamma^1 : \mathrm{int}\, \mathsf{f} \in \Gamma^1\}$.

b) Assuming that the proximity and the contiguities are Lodato and the trace filters are Cauchy, let $\Gamma^0_L$ be the contiguity on $X$ for which $\{\mathrm{int}\, \mathsf{f} : \mathsf{f} \in \Gamma^0\}$ is a base. $\diamondsuit$

Observe that $\Gamma^0_L = \Gamma(\mathsf{M}^0_L(\delta, \mathsf{M}^0(\Gamma_i)))$. A subbase for $\Gamma^0_L$ can be described similarly to 5.14 (1) – (2). If $c$ is a topology then the $c$-open covers in $\Gamma^1$ form a base for $\Gamma^1_L$.

**Lemma.** *A family of contiguities in a proximity space has a Lodato extension iff*

(i) *the proximity and the contiguities are Lodato;*

(ii) $\mathrm{int}\, \mathsf{f}^0_i$ *is a $\delta$-cover* $(i \in I, \mathsf{f}_i \in \Gamma_i)$;

(iii) $(\mathrm{int}\, \mathsf{f}^0_i)|X_j \in \Gamma_j$ $(i, j \in I, \mathsf{f}_i \in \Gamma_i)$.

*If these conditions are satisfied then $\Gamma^0_L$ is the coarsest and $\Gamma^1_L$ the finest extension.*

**Proof.** It follows from Lemmas 5.2, 5.17 and 5.16 that the conditions are necessary and sufficient, and $\Gamma_L^0$ is the coarsest extension.

Assume that $\Gamma$ is a Lodato extension, and $\mathfrak{f} \in \Gamma$. Then $\operatorname{int} \mathfrak{f} \in \Gamma$, so $\operatorname{int} \mathfrak{f} \in \Gamma^1$ by Theorem 6.2, therefore $\mathfrak{f} \in \Gamma_L^1$, i.e. $\Gamma \subset \Gamma_L^1$. This means that if $\Gamma_L^1$ is a Lodato extension then it can only be the finest one. Taking $\Gamma = \Gamma_L^0$, we have $\Gamma_L^0 \subset \Gamma_L^1$, and $\Gamma_L^1 \subset \Gamma^1$ by the definition; hence $\Gamma_L^1$ is an extension, and, being compatible, it is clearly Lodato. $\diamondsuit$

Condition (iii) is not superfluous: take the contiguities from Example 4.5, with the Euclidean proximity on $X$. Condition (ii) can be, similarly to 5.18 (a) and (b), decomposed into two parts, neither of which is sufficient in itself (although either implies that the trace filters are Cauchy, see in 5.18):

**Examples.** a) Taking $X, X_1$ from Example 4.5, with the Euclidean proximity on $X$, we modify $\Gamma_1$ by interchanging the role of the coordinates, and adding one more member to the covers in the subbase: let $\{\mathfrak{f}_1(k) : k \in \mathbb{N}\}$ be a subbase for $\Gamma_1$, where

$$\begin{aligned}\mathfrak{f}_1(k) = \{\{(1/m, 1/n) : m, n \geq k, n \not\equiv \mu(\text{mod}3)\} : \mu = 0, 1, 2\} \cup \\ \cup \{\{(1/m, 1/n) : n \geq k\} : m < k\} \cup \{\{(1/m, 1/n) : m \geq k\} : n < k\} \cup \\ \cup \{\{(1/m, 1/n)\} : m, n < k\} \cup \{\{(1/m, 1/n) : n > m \geq k\}\}.\end{aligned}$$

Now the last member in the definition of $\mathfrak{f}_1(k)$ guarantees that the condition analogous to 5.18 (a) is satisfied. But (b) fails: take $\mathfrak{c}_1 = \mathfrak{f}_1(1)$, $A = X_1^r$ and $B = \{(1/n, 1/n) : n \in \mathbb{N}\}$.

b) Let $X = (\mathbb{R}\setminus\{0\}) \times \mathbb{R}$, $X_0 = (\mathbb{R}\setminus\{0\})^2$, $\delta$ the Euclidean proximity on $X$, $S_1 = ]\leftarrow, 0[$, $S_2 = ]0, \rightarrow[$,

$$\mathfrak{e}_0 = \{X_0 \setminus (S_u \times S_v) : u = 1, 2,\ v = 1, 2\},$$

and $\Gamma^0(\delta_0) \cup \{\mathfrak{e}_0\}$ a subbase for $\Gamma_0$. $\Gamma^0(\delta_0)$ is compatible and Lodato (Lemma 5.15), and $\mathfrak{e}_0$ is a $c_0$-open $\delta_0$-cover, so $\Gamma_0$ is a compatible Lodato contiguity by Lemma 6.1. Now $\mathfrak{e}_0$, $S_1 \times \{0\}$ and $S_2 \times \{0\}$ show that (a) is not fullfilled. But (b) holds:

We may assume (by Axiom P5, and for reasons of symmetry) that $A \subset S_2 \times \{0\}$ and $B \subset (S_1 \cup S_2) \times S_2$. Take $\mathfrak{f}_0 \in \Gamma^0(\delta_0)$ such that $\mathfrak{f}_0(\cap)\mathfrak{e}_0$ refines the prescribed $\mathfrak{c}_0 \in \Gamma_0$. As $\Gamma^0(\delta)$ is a Lodato extension of $\Gamma^0(\delta_0)$, (b) holds with $\mathfrak{f}_0$ instead of $\mathfrak{c}_0$, thus we can pick $F_0 \in \mathfrak{f}_0$ and $x \in A$ such that $F_0 \in \mathfrak{s}_0(x)$ and $F_0 \cap B \neq \emptyset$. Now with $C_0 = F_0 \cap (X_0 \setminus S_1^2) \in \mathfrak{f}_0(\cap)\mathfrak{e}_0$ we have $C_0 \in \mathfrak{s}_0(x)$ and $C_0 \cap B \neq \emptyset$ (since $B \subset (X_0 \setminus S_1^2)$); hence (b) holds with $\mathfrak{f}_0(\cap)\mathfrak{e}_0$, therefore also with $\mathfrak{c}_0$. $\diamondsuit$

These examples could also have been used in 5.18, had we not made in § 5C a point of requiring that the induced contiguities and semi-uniformities should have Lodato extensions whenever possible.

**Corollary.** *A family of contiguities in a Lodato proximity space has a Lodato extension iff* $\{\Gamma_i, \Gamma_j\}$ *has a Lodato extension for any* $i, j \in I$. ◊

Compare this corollary with 5.21.

**6.9 Corollary.** *A family of contiguities in a Lodato proximity space has a Lodato extension iff it has a Lodato extension in* $(X, c)$ *and each* $\{\Gamma_i\}$ *has a Lodato extension in* $(X, \delta)$.

**Proof.** Lemma 6.8 and Theorem 4.3. ◊

**6.10 Lemma.** *Under the assumptions of Definition 6.8 b), a family of contiguities in a proximity space has a Lodato extension iff* $\Gamma_L^0 \subset \Gamma_L^1$.

**Proof.** The necessity follows from the last statement in Lemma 6.8. Conversely, assume that $\Gamma_L^0 \subset \Gamma_L^1$. It is clear from the definitions that $\Gamma_L^1 \subset \Gamma_L^0$ and $\Gamma_L^1 \subset \Gamma^1$, hence $\Gamma_L^1$ is an extension by Theorem 6.2; $\Gamma_L^1$ is Lodato, because $c$ is a topology. ◊

**6.11 Theorem.** *A family of Lodato contiguities given on closed subsets in a Lodato proximity space has Lodato extensions;* $\Gamma^0 = \Gamma_L^0$ *is the coarsest and* $\Gamma_L^1$ *the finest Lodato extension.*

**Proof.** Theorem 5.22 and Lemma 6.8. ◊

$\Gamma^0$ and $\Gamma_L^0$ can be different if the subsets are not closed: take $X$, $X_1$ and $\Gamma_1$ from Example 4.5, with the Euclidean proximity of $X$ (cf. Example 5.22). $(\Gamma^0(\delta) =)\Gamma_L^0(\delta) \neq \Gamma_L^1(\delta)$ for $\delta$ from Example 5.2: if $A$, $B$, $C \subset X$ are disjoint infinite sets then $\mathsf{f} = \{A^r, B^r, C^r\}$ is clearly a finite open $\delta$-cover, so $\mathsf{f} \in \Gamma_L^1(\delta)$; but $\mathsf{f} \notin \Gamma^0(\delta)$, since each cover $\mathsf{c}_{P,Q}$ $(P\bar{\delta}Q)$, and so each element of $\Gamma^0(\delta)$, contains at least one cofinite set. (The result cited in 6.3 could also be used, since $c$ is discrete, and so $\Gamma_L^1(\delta) = \Gamma^1(\delta)$.) In Example 6.5, $\mathsf{f}' \in \Gamma_R^1(\delta) \setminus \Gamma_L^1(\delta)$; $\Gamma_R^1(\delta)$ and $\Gamma^1(\delta)$ were different in the same example.

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