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ON k-HAMILTON GEOMETRY

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Dedicated to Professor Radu Miron.

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Abstract: The theory of Hamilton geometry $(k=1)$ has been developed by R. MIRON ([15], [16]). In this paper we study the theory of k-Hamilton geometry $(k>1)$ using Miron's theory of Hamilton geometry as a pattern. First we show the reasons for undertaking this work and the previous results in the theory of k-Lagrange geometry ([8], [9], [14]). Next, we consider the vector bundle $\xi = (\bigoplus_1^k T^*M, \pi^*, M)$ and describe the geometry of the total space $E^* = \bigoplus_1^k T^*M$ called k-Hamilton geometry.

1. Introduction

It is well-known that parameter-invariant problems (i.e. homogeneous cases) in the calculus of variations lend themselves well to geometrical interpretation and this has given rise to metric differential geometries such as that of Finsler and its special cases: Riemannian and Minkowskian geometry. But it is also known from the classical calculus of variations (c.f. [3], [17], [18]) that there exist several problems for which the fundamental integral is dependent on the choice of the parameter. This dependence implies that the Lagrangian cannot possess certain homogenity properties.

It was J. Kern [6] who introduced the term *Lagrange geometry* with a regular Lagrangian but without homogenity condition. It is obvious that this geometry is more general than the Finslerian.

Although the introduction of the notion of Lagrange geometry belongs to J. Kern, the whole theory of Lagrange geometry has been developed by Romanian geometers led by R. Miron (c.f. [1], [2], [11], [12], [13]). In the models for Lagrange geometry the basic manifold is the total space TM of the tangent bundle to a manifold M.

In a series of papers ([8], [9], [14]) *we have constructed a geometrical model for variational problems of multiple integrals called k-Lagrange geometry.* The formulation of variational problems of multiple integrals (c.f. [19], [20]) suggests that a *geometrical model* could be the *total space* $E = \bigoplus_1^k TM$ of the *vector bundle* $\bigoplus_1^k TM \to M$.

We note that this vector bundle was used by C. Günther [5] too. *Our theory,* on the contrary, *is based on the study of a metric which is derived from the Lagrangian. We have used as a pattern the geometry of the total space of a vector bundle* as it was developed by R. Miron [10].

We have described differential structures, nonlinear connections, d-connections and metrical structures on $E = \bigoplus_1^k TM$. We have pointed out that E carries several tensorial structures and studied conditions for their integrability. Furthermore we have given an application of k-Langrange geometry considering the Moór equivalence problem ([17], [18]) in the calculus of variations of multiple integrals.

In the papers [15] and [16] R. Miron has introduced a new concept: *Hamilton geometry* which corresponds to the notion of Lagrange geometry under the duality of the tangent ($TM \to M$) and the cotangent ($T^*M \to M$) bundles. He studied also its applications in theoretical physics.

This article has been inspired by R. Miron's papers mentioned above and by the theory of k-Lagrange geometry.

Let us consider the 1-jet bundle $J^1(\mathbb{R}^k, TM) \to M$ together with the 1-cojet bundle $J^1(TM,\mathbb{R}^k) \to M$. The 1-jet bundle has typical fiber $L(\mathbb{R}^k,\mathbb{R}^n)$ while the 1-cojet bundle has typical fiber $L(\mathbb{R}^n,\mathbb{R}^k)$. We recall that $J^1(\mathbb{R}^k, TM) \simeq \mathrm{Hom}(\mathbb{R}^k, TM) \simeq TM \otimes (\mathbb{R}^k)^*$ as vector bundles ([5], [8]). Moreover we have $J^1(TM,\mathbb{R}^k) \simeq \mathrm{Hom}\,(TM,\mathbb{R}^k) \simeq$ $\simeq T^*M \otimes \mathbb{R}^k$ as vector bundles too. Here $\mathrm{Hom}(\mathbb{R}^k, TM)$ denotes the total space of the vector bundle defined by all linear maps $\mathbb{R}^k \to$ $\to T_qM, q \in M$. Since there exist the isomorphisms $\mathrm{Hom}(\mathbb{R}^k, TM) \simeq$ $\simeq \bigoplus_1^k TM$ and $\mathrm{Hom}\,(TM,\mathbb{R}^k) \simeq \bigoplus_1^k T^*M$, it follows that $\mathrm{Hom}(TM,\mathbb{R}^k)$ is the dual of $\mathrm{Hom}(\mathbb{R}^k, TM)$. We shall use these isomorphisms in the sequel. Let π^* be the projection on M, i.e. $\pi^* : \bigoplus_1^k T^*M \to M$. We shall consider the vector bundle $\zeta^* = (\bigoplus_1^k T^*M,\ \pi^*,\ M)$ and the geometry of the total space $E^* = \bigoplus_1^k T^*M$.

First we describe differential structures and nonlinear connections on E^*. We define tensorial structures on E^* and give conditions for their integrability.

Moreover, we shall study d-connections, metrical structures on E^* and the Legendre transformation between E and E^*.

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2. Differential structure on $E^* = \bigoplus_1^k T^*M$

Let (U, ψ) be a local chart on M. Then $(U, \varphi^*, \mathbb{R}^{kn})$ is a bundle chart of the vector bundle ζ^* where

$$\varphi^* : (\pi^*)^1(U) \to U \times \mathbb{R}^{kn} \tag{2.1}$$

and for $\underset{(\alpha)}{\xi}_q \in \bigoplus_1^k T_q^* M \quad (q \in M, \alpha = \overline{1,k})$ we have

$$\varphi^*(\underset{(1)}{\xi}_q, \ldots, \underset{(k)}{\xi}_q) = (p_i^\alpha). \tag{2.2}$$

We can see that

$$\underset{(\alpha)}{\xi}_q = p_i^\alpha dq^i = p_1^\alpha dq^1 + \ldots + p_n^\alpha dq^n \tag{2.3}$$

which is *a linear form* for every α.

Puttig $(q^i) = \psi(q)$ we define

$$\begin{aligned} &\text{(a)} \quad h^* : (\pi^*)^1(U) \to \psi(U) \times \mathbb{R}^{kn} \\ &\text{(b)} \quad h^*(\underset{(1)}{\xi}_q, \ldots, \underset{(k)}{\xi}_q) = (q^i, p_i^\alpha) \end{aligned} \tag{2.4}$$

then we get the canonical coordinates (q^i, p_i^α) on $(\pi^*)^1(U)$. The set of charts $((\pi^*)^1(U), h^*)$ defines a differentiable atlas on $E^* = \bigoplus_1^k T^*M$.

The transition maps on E^* are as follows:

$$\begin{aligned} &\text{(a)} \quad \bar{q}^i = \bar{q}^i(q^1, \ldots, q^n) \\ &\text{(b)} \quad \bar{p}_i^\alpha = (\bar{\partial}_i q^j) p_j^\alpha \end{aligned} \tag{2.5}$$

where $\bar{\partial}_i := \partial/\partial\bar{q}^i$. The transformation law shows that (p_i^α) can be considered as a *covariant* vector. (In the following we denote p_i^α by p_a where $\binom{\alpha}{i} := a$ and use a shorter notation $a, b, c\ldots$ instead of double covariant indices $\binom{\alpha}{i}$ or contravariant indices $\binom{i}{\alpha}$ if the computation allows it for us.)

A local natural basis of the tangent space $T_{u^*}(E^*)$ in $u^* \in E^*$ is $(\partial_i, \partial_\alpha^i) = (\partial_i, \partial^a)$ where $\partial_i := \partial/\partial_q^i$ and $\partial_\alpha^i := \partial/\partial p_i^\alpha$. Its dual basis is $(dq^i, dp_i^\alpha) := (dq^i, dp_a)$. Under a change of coordinates in (2.5) we obtain

$$\begin{aligned} &\text{(a)} \ \bar{\partial}_i = \bar{\partial}_i q^j \partial_j - p_k^\alpha \bar{\partial}_i \bar{\partial}_j q^k \partial_h \bar{q}^j \partial_\alpha^h \\ &\text{(b)} \ \bar{\partial}_\alpha^i = \partial_j \bar{q}^i \partial_\alpha^j \qquad (\bar{\partial}_\alpha^i := \partial/\partial\bar{p}_i^\alpha) \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &\text{(a)} \ d\bar{q}^j = \partial_i \bar{q}^j dq^i \\ &\text{(b)} \ d\bar{p}_i^\alpha = p_k^\alpha \bar{\partial}_i \bar{\partial}_j q^k \partial_h \bar{q}^j dq^h + \bar{\partial}_i q^j dp_j^\alpha, \end{aligned} \tag{2.7}$$

respectively.

Hence we have a generalization of a result given by R. Miron in the case $k = 1$ in [15]:

Proposition 2.1. *Setting*

$$\widetilde{\mathrm{p}} = (\overset{1}{p}, \ldots, \overset{k}{p}) \tag{2.8}$$

we get an $\mathbb{R}^k$-valued 1-form on E^* whose components

$$\overset{\alpha}{p} = p_i^\alpha dq^i \qquad (\alpha = \overline{1,k})$$

will be called fundamental forms.

The differential of $\widetilde{\mathrm{p}}$ is obtained by differentiating its components. So we have the following $\mathbb{R}^k$-valued 2-form on E^*:

$$\omega := d\widetilde{\mathrm{p}} \tag{2.9}$$

where

$$\begin{aligned} &\text{(a)} \quad \omega = (\overset{1}{\omega}, \ldots, \overset{k}{\omega}), \\ &\text{(b)} \quad \overset{\alpha}{\omega} = dp_i^\alpha \wedge dq^i \qquad (\alpha = \overline{1,k}). \end{aligned} \tag{2.10}$$

Any 2-form $\overset{\alpha}{\omega}$ is nondegenerate and $d\overset{\alpha}{\omega} = 0$, hence

$$d\omega = (d\overset{1}{\omega}, \ldots, d\overset{k}{\omega}) = 0. \tag{2.11}$$

This means that the $\mathbb{R}^k$-valued 2-form ω is closed. Therefore ω is a *polysymplectic form* and (E^*, ω) is a *polysymplectic manifold* in C. Günther's sense [5].

3. Nonlinear connections on $\mathbf{E}^* = \bigoplus_1^k \mathbf{T}^*\mathbf{M}$

The kernel of $D\pi^*$ (the differential of π^*) is a subbundle of the cotangent bundle $\bigoplus_1^k T^*M \to M$. It will be denoted by $VE^* \to E^*$ and will be called the vertical bundle. A map $u^* \to V_{u^*}(E^*)$ where $u^* \in E^*$ and $V_{u^*}(E^*)$ is the fiber of the vertical bundle, defines a distribution $\mathbf{V}$ on $E^* = \bigoplus_1^k T^*M$ which will be called the vertical distribution.

Definition 3.1. A *nonlinear connection* on E^* is a distribution $\mathbf{N}$: $u^* \in E^* \to N_{u^*} \subset T_{u^*}(E^*)$ which is supplementary to the vertical distribution $\mathbf{V}$, i.e.

$$T_{u^*}(E^*) = N_{u^*} \oplus V_{u^*} \tag{3.1}$$

holds for any $u^* \in E^*$.

The vertical subspaces $V_{u^*}(E^*)$ are spanned by $(\partial^i_\alpha) = (\partial^a)$. The *horizontal distribution* $\mathbf{N}$ is locally determined by

$$\delta^*_i = \partial_i + N^\alpha_{ji}(q,p)\partial^j_\alpha \qquad (\delta^*_i := \delta/\delta q^i). \tag{3.2}$$

Hence $(\delta^*_i, \partial^i_\alpha) = (\delta^*_i, \partial^a)$ is a local frame, adapted to the decomposition of $T_{u^*}(E^*)$. The real functions $N^\alpha_{ji}(q,p)$ defined on $(\pi^*)^{-1}(U)$ are called the local coefficients of the nonlinear connection $\mathbf{N}$ and characterize it. The dual frame is $(dq^i, \delta^* p^\alpha_i) = (dq^i, \delta^* p_a)$ where

$$\delta^* p_a = dp_a - N_{aj}dq^j. \tag{3.3}$$

Under a change of coordinates in (2.5) we get their transformation laws:

$$\begin{aligned} &\text{(a)}\ \ \bar\delta^*_j = \bar\partial_j q^i \delta^*_i && \text{(b)}\ \ \bar\partial^i_\alpha = \partial_j \bar q^i \partial^j_\alpha \\ &\text{(c)}\ \ d\bar q^i = \partial_j \bar q^i dq^j && \text{(d)}\ \ \delta^* \bar p^\alpha_i = \bar\partial_i q^j \delta^* p^\alpha_j . \end{aligned} \tag{3.4}$$

With respect to the transformation (2.5) the coefficients $N^\alpha_{ji}(q^s, p^\beta_s)$ of a nonlinear connection $\mathbf{N}$ have the following transformation law:

$$\bar N^\alpha_{ji}(\bar q, \bar p) = \bar\partial_j q^k \bar\partial_i q^h N^\alpha_{kh}(q,p) + p^\alpha_k \bar\partial_j \bar\partial_i q^k \tag{3.5}$$

for every α.

A direct calculation gives

$$\begin{aligned} &\text{(a)}\quad [\delta^*_i, \delta^*_j] = (\delta^*_i N_{aj} - \delta^*_j N_{ai})\partial^a \\ &\text{(b)}\quad [\delta^*_i, \partial^a] = -(\partial^a N_{bi})\partial^b \\ &\text{(c)}\quad [\partial^a, \partial^b] = 0. \end{aligned} \tag{3.6}$$

We can associate to a nonlinear connection on E^* the following geometrical objects:

$$\begin{aligned} &\text{(a)}\qquad \tau_{ia} = N_{ia} - N_{ai} && (a := \tbinom{\alpha}{j}), \\ &\text{(b)}\qquad R_{aij} = \delta^*_i N_{aj} - \delta^*_j N_{ai} && (a := \tbinom{\alpha}{k}). \end{aligned} \tag{3.7}$$

They give us antisymmetric d-tensor fields in i and j. Moreover we obtain

Proposition 3.1. *The horizontal distribution* $\mathbf{N}$ *is integrable iff* $R_{aij} = 0$.

The decomposition of the tangent space $T_{u^*}(E^*)$ at the point $u^* \in E^*$ is the following

$$T_{u^*}(E^*) = H_{u^*}(E^*) \oplus V_{u^*}(E^*). \tag{3.8}$$

This decomposition defines a decomposition of the cotangent space $T^*_{u^*}(E^*)$ in $u^* \in E^*$:

$$T^*_{u^*}(E^*) = H^{\perp}_{u^*}(E^*) \oplus V^{\perp}_{u^*}(E^*). \tag{3.9}$$

The coframe $(dq^i, \delta^* p_i^\alpha)$ is adapted to this decomposition.

The elements of $H^{\perp}_{u^*}(E^*)$ are 1-forms which vanish for vertical vector fields and the element of $V^{\perp}_{u^*}(E^*)$ are also 1-forms which vanish for horizontal vector fields.

An easy computation shows that dq^i are *horizontal 1-forms* and $\delta^* p_a$ are *vertical 1-forms.*

By using $\delta^* p_i^\alpha$ we find that

$$\overset{\alpha}{\omega}{}^{\perp} = \tfrac{1}{2}\tau^{\alpha}_{ji} dq^j \wedge dq^i + \delta^* p_i^\alpha \wedge dq^i \qquad (\alpha = \overline{1,k}) \tag{3.11}$$

which is compatible with the decomposition (3.9). This formula introduces the 2-forms

$$\overset{\alpha}{\Theta} = \delta^* p_i^\alpha \wedge dq^i \qquad (\alpha = \overline{1,k}) \tag{3.12}$$

considered by R. Miron [15] in the case $k = 1$.

The $\mathbb{R}^k$-valued 2-form $\Theta = (\overset{1}{\Theta}, \dots, \overset{k}{\Theta})$ defines an *almost polysymplectic structure* on E^*. As we have seen above this is a polysymplectic form if $\tau^{\alpha}_{ji} = 0$ for every α. In this case the nonlinear connection $\mathbf{N}(N^{\alpha}_{ji})$ is symmetric.

As in the case of *Hamilton geometry* (for $k = 1$) we have the following relations between the adapted frames on $T_{u^*}(E^*)$ and $T^*_{u^*}(E^*)$ respectively:

$$\begin{array}{llll} \text{(a)} & < dq^i, \delta^*_j >= \delta^i_j & \text{(b)} & < \delta^* p^\alpha_i, \delta^*_j >= 0 \\ \text{(c)} & < dq^i, \partial^j_\alpha >= 0 & \text{(d)} & < \delta^* p^\alpha_i, \partial^j_\beta >= \delta^j_i \delta^\alpha_\beta . \end{array} \tag{3.13}$$

We can associate to $\mathbf{N}$ an *almost product structure* $\mathbf{P}$ on E^* defined as follows:

$$\text{(a)}\ \mathbf{P}(\delta^*_i) = \delta^*_i \qquad \text{(b)}\ \mathbf{P}(\partial^i_\alpha) = -\partial^i_\alpha. \tag{3.14}$$

It is easy to check that

$$\overset{\alpha}{\Theta}(PX, PY) = -\overset{\alpha}{\Theta}(X, Y) \qquad (\alpha = \overline{1,k}) \tag{3.15}$$

for any $X, Y \in \mathcal{X}(E^*)$. This can be written as

$$\Theta(PX, PY) = -\Theta(X, Y) \tag{3.16}$$

where $\Theta = (\overset{1}{\Theta}, \dots, \overset{k}{\Theta})$. We shall call $(\Theta, \mathbf{P})$ *almost hyperbolic structure* on E^*.

4. Tensorial structures on $\mathbf{E}^* = \overset{k}{\underset{1}{\bigoplus}} \mathbf{T}^*\mathbf{M}$

If we set

(4.1) (a) $\overset{\alpha}{F}(\delta_i^*) = -\partial_\alpha^i$ (b) $\overset{\alpha}{F}(\partial_\alpha^i) = \delta_i^*$ (c) $\overset{\alpha}{F}(\partial_\beta^i) = 0, (\forall \beta \neq \alpha)$

we obtain k *f-structures* for which

$$\overset{\alpha}{F}{}^3 + \overset{\alpha}{F} = 0 \qquad (\alpha = \overline{1,k}). \tag{4.2}$$

Analogously, we can define as in the case of *k-Lagrange geometry [8]* the following *tensorial structures* $\overset{\alpha}{Q}(\alpha = \overline{1,k})$

(4.3) (a) $\overset{\alpha}{Q}(\delta_i^*) = \partial_\alpha^i$ (b) $\overset{\alpha}{Q}(\partial_\alpha^i) = \delta_i^*$ (c) $\overset{\alpha}{Q}(\partial_\beta^i) = 0$ $(\forall \beta \neq \alpha)$

and we obtain

$$\overset{\alpha}{Q}{}^3 - \overset{\alpha}{Q} = 0 \qquad (\alpha = \overline{1,k}). \tag{4.4}$$

Moreover we have

$$\begin{aligned} &\text{(a)} \quad \overset{\alpha}{\Theta}(\overset{\beta}{F}X, \overset{\beta}{F}Y) = \overset{\alpha}{\Theta}(X, Y), \\ &\text{(b)} \quad \overset{\alpha}{\Theta}(\overset{\beta}{Q}X, \overset{\beta}{Q}Y) = \overset{\alpha}{\Theta}(X, Y) \end{aligned} \tag{4.5}$$

for any α, β and any $X, Y \in \mathcal{X}(E^*)$.

Now we study the *integrability of the structures* $\overset{\alpha}{F}$ and $\overset{\alpha}{Q}$ respectively.

We have the following conditions

$$(4.6)\qquad \begin{array}{ll}(a) & \operatorname{rank}(\overset{\alpha}{F}) = n < nk \\ (b) & \operatorname{rank}(\overset{\alpha}{Q}) = n < nk\end{array} \qquad (k > 1)$$

It is easy to see that $F_1 = -\overset{\alpha}{F}{}^2$ and $F_2 = \overset{\alpha}{F}{}^2 + I$ are two supplementary projectors associated to $\overset{\alpha}{F}$. It is said that $\overset{\alpha}{F}$ is integrable if the distributions associated to F_1 and F_2 are integrable. As V. Duc [4] proved these distributions are integrable iff $N_{\overset{\alpha}{F}{}^2} = 0$ where $N_{\overset{\alpha}{F}{}^2}$ means the Nijenhuis tensor field of $\overset{\alpha}{F}{}^2$ $(\alpha = \overline{1,k})$.

Further we associate with $\overset{\alpha}{Q}$ the set of projectors $Q_1 = I - \overset{\alpha}{Q}{}^2$, $Q_2 =$
$= \frac{1}{2}(\overset{\alpha}{Q} + \overset{\alpha}{Q}{}^2)$ and $Q_3 = \frac{1}{2}(-\overset{\alpha}{Q} + \overset{\alpha}{Q}{}^2)$. Let $D_i (i = 1,2,3)$ be the distributions defined by these projectors. The structure $\overset{\alpha}{Q}$ is said to be integrable if the distributions D_i and $D_i + D_j (j = 1,2,3)$ are integrable. V. Duc [4] proved that $\overset{\alpha}{Q}$ is integrable iff $N_{\overset{\alpha}{Q}} = O$ where $N_{\overset{\alpha}{Q}}$ is the Nijenhuis tensor field of $\overset{\alpha}{Q} (\alpha = \overline{1,k})$.

Using the definition of $\overset{\alpha}{F}$ we get

$$(4.7)\qquad (a)\ \overset{\alpha}{F}{}^2(\delta_i^*) = -\delta_i^* \quad (b)\ \overset{\alpha}{F}{}^2(\partial_\alpha^i) = -\partial_\alpha^i \quad (c)\ \overset{\alpha}{F}{}^2(\partial_\beta^i) = 0 \quad (\beta \neq \alpha).$$

The relation (4.2) implies that

$$\overset{\alpha}{F}{}^4 = -\overset{\alpha}{F}{}^2.$$

Hence we have

$$(4.9)\qquad N_{\overset{\alpha}{F}{}^2}(X,Y) = [\overset{\alpha}{F}{}^2 X, \overset{\alpha}{F}{}^2 Y] + \overset{\alpha}{F}{}^4[X,Y] - \overset{\alpha}{F}{}^2[\overset{\alpha}{F}{}^2 X, Y] -$$
$$-\overset{\alpha}{F}{}^2[X, \overset{\alpha}{F}{}^2 Y] = [\overset{\alpha}{F}{}^2 X, \overset{\alpha}{F}{}^2 Y] - \overset{\alpha}{F}{}^2[X,Y] - \overset{\alpha}{F}{}^2[\overset{\alpha}{F}{}^2 X, Y] - \overset{\alpha}{F}{}^2[X, \overset{\alpha}{F}{}^2 Y].$$

To find conditions for the integrability of $\overset{\alpha}{F}$ which are equivalent to V. Duc's conditions we shall compute $N_{\overset{\alpha}{F}{}^2}$ in the adapted frame $(\delta_i^*, \partial_\alpha^i)$ using the relations (3.6) and (3.7) (b). We obtain for fixed α

$$(4.10)\qquad \begin{array}{lll}(a) & N_{\overset{\alpha}{F}{}^2}(\delta_j^*, \delta_k^*) = R_{ijk}^{\beta} \partial_\beta^i & (\text{summing over } \beta \neq \alpha) \\ (b) & N_{\overset{\alpha}{F}{}^2}(\delta_j^*, \partial_\alpha^k) = -\partial_\alpha^k (N_{ij}^{\beta}) \partial_\beta^i & (\text{summing over } \beta \neq \alpha) \\ (c) & N_{\overset{\alpha}{F}{}^2}(\delta_i^*, \partial_\beta^k) = 0 & (\beta \neq \alpha)\end{array}$$

(d) $N_{\overset{\alpha}{F}^2}(\partial^j_\alpha, \partial^k_\alpha) = 0$

(e) $N_{\overset{\alpha}{F}^2}(\partial^j_\alpha, \partial^k_\beta) = 0 \qquad (\beta \neq \alpha)$

(f) $N_{\overset{\alpha}{F}^2}(\partial^j_\beta, \partial^k_\gamma) = 0 \qquad (\beta \neq \alpha,\ \gamma \neq \alpha).$

Theorem 4.1. $\overset{\alpha}{F}$ *is integrable iff*

(4.11) (a) $R^\beta_{ijk} = 0 \quad (\forall \beta \neq \alpha)$, (b) $\partial^k_\alpha(N^\beta_{ij}) = 0 \quad (\forall \beta \neq \alpha)$.

Corollary 4.1. *All* $\overset{\alpha}{F}(\alpha = \overline{1,k})$ *are integrable iff* R^α_{ijk} *vanish and* N^α_{ik} *depend only on* $(p^\alpha_i)(\alpha = \overline{1,k})$.

Remark. Condition (4.11) (a) shows that integrability of the horizontal distribution is a necessary and sufficient condition for the integrability of $\overset{\alpha}{F}$.

Now we proceed similarly for $\overset{\alpha}{Q}$ using its definition. In general we have

$$(4.13) \qquad N_{\overset{\alpha}{Q}}(X,Y) = [\overset{\alpha}{Q}X, \overset{\alpha}{Q}Y] + \overset{\alpha}{Q}{}^2[X,Y] - \overset{\alpha}{Q}[\overset{\alpha}{Q}X, Y] - \\ -\overset{\alpha}{Q}[X, \overset{\alpha}{Q}Y], \quad X, Y \in \mathcal{X}(E^*),$$

hence for the adapted frame we obtain

$$\text{(a)} \quad N_{\overset{\alpha}{Q}}(\delta^*_j, \delta^*_k) = R^\alpha_{ijk}\partial^i_\alpha - \sum_i(\partial^k_\alpha N^\alpha_{ij} - \partial^j_\alpha N^\alpha_{ik})\delta^*_i$$

(not summing over α)

$$\text{(b)} \quad N_{\overset{\alpha}{Q}}(\delta^*_j, \partial^k_\alpha) = \partial^j_\alpha N^\gamma_{ik}\partial^i_\gamma - \partial^k_\alpha N^\alpha_{ij}\partial^i_\alpha - \sum_i R^\alpha_{ijk}\delta^*_i = \\ = \partial^j_\alpha N^\gamma_{ik}\partial^i_\gamma - \sum_i R^\alpha_{ijk}\delta^*_i \quad (\gamma \neq \alpha)$$

$$(4.14)\ \text{(c)} \quad N_{\overset{\alpha}{Q}}(\delta^*_j, \partial^k_\beta) = -\sum_i(\partial^k_\beta N^\alpha_{ij})\delta^*_i \quad (\beta \neq \alpha)$$

$$\text{(d)} \quad N_{\overset{\alpha}{Q}}(\partial^j_\alpha, \partial^k_\alpha) = R^\gamma_{ijk}\partial^i_\gamma + \sum_i(\partial^k_\alpha N^\alpha_{ij} - \partial^j_\alpha N^\alpha_{ik})\delta^*_i$$

(not summing over α)

$$\text{(e)} \quad N_{\overset{\alpha}{Q}}(\partial^j_\alpha, \partial^k_\beta) = \sum_i \partial^k_\beta N^\alpha_{ij}\delta^*_i \quad (\beta \neq \alpha)$$

$$\text{(f)} \quad N_{\overset{\alpha}{Q}}\ \partial^j_\beta, \partial^k_\gamma) = 0 \quad (\beta \neq \alpha, \gamma \neq \alpha).$$

So we have proved

Theorem 4.2. $\overset{\alpha}{Q}$ *is integrable iff*

$$(4.15)\quad \begin{aligned} &(a)\ R^{\alpha}_{ijk} = 0 \ (\textit{i.e. the horizontal distribution is integrable})\\ &(b)\ \partial^j_\alpha N^\gamma_{ik} = 0 \qquad (\gamma \neq \alpha)\\ &(c)\ \partial^k_\alpha N^\alpha_{ij} = \partial^j_\alpha N^\alpha_{ik}\\ &(d)\ \partial^k_\beta N^\alpha_{ij} = 0 \qquad (\beta \neq \alpha). \end{aligned}$$

Corollary 4.2. *All* $\overset{\alpha}{Q}(\alpha = \overline{1,k})$ *are integrable iff*

$$(4.16)\quad \begin{aligned} &(a)\ R^{\alpha}_{ijk} = 0\\ &(b)\ N^\alpha_{ij} \textit{ depends only on } (p^\alpha_i)(\alpha \textit{ is fixed}). \end{aligned}$$

We can see that the condition (4.14) (c) is equivalent to (4.15) (a) and the condition (4.15) (d) follows from (4.15) (b) and (c).

Corollary 4.3. *If* $\overset{\alpha}{Q}$ *is integrable then* $\overset{\alpha}{F}$ *is integrable. If all* $\overset{\alpha}{Q}$ *are integrable then all* $\overset{\alpha}{F}$ *are integrable.*

5. *d*-connections on $E^* = \bigoplus_1^k T^*M$

A *distinguished connection* – shortly *d-connection* – on E^*, endowed with a nonlinear connection, is a linear connection D on E^* which preserves by parallel displacements the horizontal and the vertical distributions.

Now we are interested in its *local* representation. We put as usual:

$$(5.1)\quad (a)\ \ D^h_X Y = D_{hX} Y \quad (b)\ \ D^\nu_X = D_{\nu X} Y \quad (X, Y \in \mathcal{X}(E^*))$$

and with respect to the adapted frame $(\delta^*_i, \partial^i_\alpha)$ we set

$$(5.2)\quad \begin{aligned} &(a)\ \ D^h_{\delta^*_k}\delta^*_j = \overset{*}{L}{}^i_{jk}\delta^*_i &&(b)\ \ D^h_{\delta^*_k}\partial^j_\beta = \overset{*}{L}{}^{j\alpha}_{\beta ik}\partial^i_\alpha\\ &(c)\ \ D^\nu_{\partial^k_\beta}\delta^*_j = \overset{*}{C}{}_j{}^i{}_\beta{}^k\delta^*_i &&(d)\ \ D^\nu_{\partial^k_\beta}\partial^j_\gamma = \overset{*}{C}{}^{\alpha jk}_{i\gamma\beta}\partial^i_\alpha. \end{aligned}$$

Hence we have obtained a set of functions

$$(5.3) \qquad \overset{*}{\Gamma}D = (\overset{*}{L}{}^{i}_{jk}(q,p),\ \overset{*}{L}{}^{j\alpha}_{\beta ik}(q,p),\ \overset{*}{C}{}_{j}{}^{i}{}_{\beta}^{k}(q,p),\ \overset{*}{C}{}^{\alpha jk}_{i\gamma\beta}(q,p)).$$

The set $\overset{*}{L}{}^{i}_{jk}$ changes like the components of a linear connection on M and the set $\overset{*}{L}{}^{j\alpha}_{\beta ik}$ changes like the components of a linear connection in a vector bundle if we consider $\binom{j}{\beta}$, $\binom{\alpha}{i}$ as contravariant and covariant indices ([8]). The $\overset{*}{C}{}_{j}{}^{i}{}_{\beta}^{k}$ and $\overset{*}{C}{}^{\alpha jk}_{i\gamma\beta}$ change like the components of the d-tensor fields on E^*.

The set $\overset{*}{\Gamma}D$ characterizes a d-linear connection, i.e. if $\overset{*}{\Gamma}D$ is given, there exists a unique d-connection such that its local coefficients are just $\overset{*}{\Gamma}D$.

Now the h- and ν-*covariant derivatives* can also be considered with respect to $\overset{*}{\Gamma}D$. Since later we need the h- and ν-covariant derivatives of a double covariant tensor field $g = g_{ij}(q,p)dq^i \otimes dq^j$ on M and of a double contravariant tensor field $\widetilde{g} = g^{ij}_{\alpha\beta}(q,p)\delta^* p^{\alpha}_{i} \otimes \delta^* p^{\beta}_{j}$ on E^* now we give them

$$(5.4) \qquad \begin{aligned} &\text{(a)} \quad g_{ijk} = \delta^*_k g_{ij} - \overset{*}{L}{}^{s}_{ik} g_{sj} - \overset{*}{L}{}^{s}_{jk} g_{is},\\ &\text{(b)} \quad g_{ij\alpha}{}^{k} = \partial^k_\alpha g_{ij} - \overset{*}{C}{}^{sk}_{i\alpha} g_{sj} - \overset{*}{C}{}^{sk}_{j\alpha} g_{is},\\ &\text{(c)} \quad g^{i\ j}_{\alpha\beta k} = \delta^*_k g^{i\ j}_{\alpha\beta} + \overset{*}{L}{}^{i\ \gamma}_{\alpha mk} g^{mj}_{\gamma\beta} + \overset{*}{L}{}^{j\ \gamma}_{\beta mk} g^{im}_{\alpha\gamma},\\ &\text{(d)} \quad g^{i\ j\ k}_{\alpha\beta\gamma} = \partial^k_\beta g^{i\ j}_{\alpha\beta} + \overset{*}{C}{}^{\epsilon\ i\ k}_{m\alpha\gamma} g^{im}_{\alpha\epsilon} + \overset{*}{C}{}^{\epsilon\ i\ k}_{m\beta\gamma} g^{im}_{\alpha\epsilon}. \end{aligned}$$

The vector field $Z = p^{\alpha}_{i}\partial^{i}_{\alpha}$ which is globally defined on E^* will be called the *Liouville* vector field on E^*.

A d-connection is of *Cartan* type if

$$(5.5) \qquad D^h_X Z = 0, \qquad D^\nu_X Z = X \qquad (\forall X \in \mathcal{X}(E^*)).$$

Expressing locally this condition we obtain:

Theorem 5.1. *A d-connection is of Cartan type iff*

$$(5.6) \qquad D^{\alpha}_{ik} = -p^{\beta}_{j}\overset{*}{L}{}^{j\alpha}_{\beta ik} + N^{\alpha}_{ik} = 0, \qquad p^{\gamma}_{j}\overset{*}{C}{}^{\alpha jk}_{i\gamma\beta} = 0.$$

The tensor field D^{α}_{ik} is called *h-deflection* tensor field associated to D. (cf. [15]).

The *torsions* of D are defined as usual. Their local coefficients are the following:

$$(5.7)\quad \begin{array}{llll} \text{(a)} & \overset{*}{T}{}^{i}_{kj} = \overset{*}{L}{}^{i}_{kj} - \overset{*}{L}{}^{i}_{jk} & \text{(b)} & \overset{*}{R}{}^{\alpha}_{ijk} = \delta^{*}_{j}(N^{\alpha}_{ik}) - \delta^{*}_{k}(N^{\alpha}_{ij}) \\ \text{(c)} & \overset{*}{C}{}_{j}{}^{i}{}^{k}_{\beta} & \text{(d)} & \overset{*}{P}{}^{\alpha}{}_{ik\beta}^{j} = \partial^{j}_{\beta} N^{\alpha}_{ik} - \overset{*}{L}{}^{j\alpha}_{\beta ik}, \\ \text{(e)} & \overset{*}{S}{}^{\alpha jk}_{i\gamma\beta} = \overset{*}{C}{}^{\alpha jk}_{i\gamma\beta} - \overset{*}{C}{}^{\alpha kj}_{i\beta\gamma}. & & \end{array}$$

6. Metrical structures on $E^{*} = \bigoplus\limits_{1}^{k} T^{*}M$

Definition 6.1. A function $\mathbf{H} : \bigoplus\limits_{1}^{k} T^{*}M \rightarrow \mathbb{R}$ is called a *Hamilton function* (or a *Hamiltonian*). If $\mathbf{H} : (q^{i}, p^{\alpha}_{i}) \rightarrow H(q^{i}, p^{\alpha}_{i})$ and the matrix with the elements

$$(6.1)\qquad g^{i\,j}_{\alpha\beta} = \partial^{i}_{\alpha}\partial^{i}_{\beta}\,\mathbf{H} \qquad (g^{i\,j}_{\alpha\beta} = g^{j\,i}_{\beta\alpha})$$

is nondegenerate, i.e. its rank is nk, then the *Hamiltonian* **H** will be called *regular.*

Theorem 6.1. *Any regular Hamiltonian* $\mathbf{H}(q,p)$ *defines a metrical structure called Hamilton structure in the vertical bundle* VE^{*}.

Proof. Define the map

$$(6.2)\quad \begin{array}{ll} \text{(a)} & g_{u^{*}} : V_{u^{*}}(E^{*}) \times V_{u^{*}}(E^{*}) \rightarrow \mathbb{R} \quad (u^{*} \in E^{*}) \text{ as} \\ \text{(b)} & g_{u^{*}}(\mathbf{X}, \mathbf{Y}) = g^{i\,j}_{\alpha\beta} X^{\alpha}_{i} Y^{\beta}_{j} \text{ where} \\ \text{(c)} & \mathbf{X} = X^{\alpha}_{i}\partial^{i}_{\alpha} \text{ and } \mathbf{Y} = Y^{\beta}_{j}\partial^{j}_{\beta} \end{array}$$

are vertical vector fields. This map is well-defined and obviously linear with respect to **X** and **Y**. By Definition 6.1. it is nondegenerate. ◊

Now if $\mathbf{g} = g_{ij}(q,p)dq^{i} \otimes dq^{j}$ is a tensor field on E^{*} such that $\det\|g_{ij}\| \neq 0$ the following metrical structure can be considered on E^{*}:

$$(6.3)\qquad \mathbf{G} = g_{ij}(q,p)dq^{i} \otimes dq^{j} + g^{i\,j}_{\alpha\beta}(q,p)\delta^{*}p^{\alpha}_{i} \otimes \delta^{*}p^{\beta}_{j}.$$

As usual we say that a d-connection is *metrical* with respect to G if

$$D_X G = 0 \tag{6.4}$$

for any $X \in \mathcal{X}(E^*)$. This condition is equivalent to the following: (6.5) $g_{ij|k} = 0, \quad g_{ij}|^k = 0, \quad g^{i\ j}_{\alpha\beta|k} = 0, \quad g^{i\ j}_{\alpha\beta}|^k_\gamma = 0.$

We can prove by direct calculation using (5.4) and the symmetric property of the Hamilton structure:

Theorem 6.2. *The following d-connection is metrical and it torsions T and S vanish:*

(6.3)

$$\text{(a) } \overset{*}{L}{}^i_{jk} = \frac{1}{2} g^{im}(\delta^*_j g_{km} + \delta^*_k g_{jm} - \delta^*_m g_{jk})$$

$$\text{(b) } \overset{*}{L}{}^{j\alpha}_{\beta ik} = \partial^j_\beta N^\alpha_{ik} + \frac{1}{2} g^{\alpha\gamma}_{im}(\delta^*_k g^{jm}_{\beta\gamma} - \partial^j_\beta(N^\epsilon_{rk}) g^{rm}_{\epsilon\gamma} - \partial^m_\gamma(N^\epsilon_{rk}) g^{rj}_{\epsilon\beta})$$

$$\text{(c) } \overset{*}{C}{}_i{}^{ik}_{\ \beta} = \frac{1}{2} g^{is}\partial^k_\beta(g_{js})$$

$$\text{(d) } \overset{*}{C}{}^{\alpha jk}_{i\gamma\beta} = \frac{1}{2} g^{\alpha\epsilon}_{ir}(\partial^j_\gamma g^{kr}_{\beta\epsilon} + \partial^k_\beta g^{jr}_{\gamma\epsilon} - \partial^r_\epsilon g^{jk}_{\gamma\beta}).$$

Here g^{ij} is the inverse of g_{jk} and $g^{\alpha\gamma}_{im}$ is the inverse of $g^{mj}_{\gamma\beta}$ i.e.

$$g^{ij} g_{jk} = \delta^i_k \quad \text{and} \quad g^{\alpha\gamma}_{im} g^{mj}_{\gamma\beta} = \delta^j_i \delta^\alpha_\beta := \delta^{\alpha j}_{i\beta}$$

hold.

An interesting particular case is obtained when g_{ij} do not depend on p. In such a case $g_{ij}(q)$ can be thought as defining a metrical structure on M and we have

$$\text{(a) } \overset{*}{L}{}^i_{jk} = \frac{1}{2} g^{im}(\delta^*_j g_{km} + \delta^*_k g_{jm} - \delta^*_m g_{jk})$$

$$\text{(b) } \overset{*}{C}{}_j{}^{i\ k}_{\ \beta} = 0 \tag{6.7}$$

$$\text{(c) } \overset{*}{L}{}^{j\alpha}_{\beta ik} = \partial^j_\beta N^\alpha_{ik} + \frac{1}{2} g^{\alpha\gamma}_{im}(\delta^*_k g^{jm}_{\beta\gamma} - \partial^j_\beta(N^\epsilon_{rk}) g^{rm}_{\epsilon\gamma} - \partial^m_\gamma(N^\epsilon_{rk}) g^{rj}_{\epsilon\beta}$$

$$\text{(d) } \overset{*}{C}{}^{\alpha jk}_{i\gamma\beta} = \frac{1}{2} g^{\alpha\epsilon}_{in}(\partial^j_r g^{kr}_{\beta\epsilon} + \partial^k_\beta g^{jr}_{\gamma\epsilon} - \partial^r_\epsilon g^{jk}_{\gamma\beta}).$$

Furthermore taking into account the Hamilton metric in (6.1) we get

$$\overset{*}{C}{}^{\alpha jk}_{i\gamma\beta} = \frac{1}{2} g^{\alpha\epsilon}_{i\ r} \frac{\partial^3 \mathbf{H}}{\partial p^\gamma_j \partial p^\beta_k \partial p^\epsilon_r} \tag{6.8}$$

and so the contravariant part of $\overset{*}{C}$

$$\overset{*}{C}{}^{jmk}_{\gamma\delta\beta} := g^{mi}_{\delta\alpha}\overset{*}{C}{}^{j\alpha k}_{\gamma i\beta} = \frac{1}{2}\frac{\partial^3 \mathbf{H}}{\partial p^\delta_m \partial p^\gamma_j \partial p^\beta_k} \tag{6.9}$$

is symmetric in the indices $\binom{j}{\gamma}$, $\binom{m}{\delta}$, $\binom{k}{\beta}$.

Remark. $\overset{*}{C}{}^{jmk}_{\gamma\delta\beta}$ corresponds to the tensor $\widetilde{C}^{jmk}$ from the Hamilton geometry ($k = 1$).

7. Legendre transformation

Let us consider $E = J^1(\mathbb{R}^k, TM) \simeq \bigoplus_1^k TM \xrightarrow{\pi} M$ and $\overset{*}{E} = J^1(TM, \mathbb{R}^k) \simeq \bigoplus_1^k T^*M \xrightarrow{\pi^*} M$. A Lagrangian $\mathbf{L}$ is a real-valued function on E (c.f. [14]). The vertical derivative of $\mathbf{L}$ is written as $d_v\mathbf{L}|_u = d(\mathbf{L}|_{E_{\pi(u)}})$ where d means differential of functions. This is a vertical 1-form because $\mathbf{L}|_{E_{\pi(u)}}$ means locally that $(x^1, \ldots, x^n)$ are fixed so $d(\mathbf{L}|_{E_{\pi(u)}}) = \partial^\alpha_i \mathbf{L} dy^i_\alpha$, i.e. an element $\partial^\alpha_i(\mathbf{L})$ of E^* is obtained. Hence a map $\mathcal{L} : E \to E^*$ can be defined as follows:

$$\mathcal{L}(x^i, y^i_\alpha) = (q^i, p^\alpha_i = \partial^\alpha_i \mathbf{L}(x, y)) \qquad ((x^i) = (q^i) \in M). \tag{7.1}$$

The map $\mathcal{L}$ is called *Legendre* map. Generally it is not a bundle morphism but it preserves the fibers.

Definition 7.1. The Lagrangian $\mathbf{L}$ is said to be *regular* if $\mathcal{L}$ is a local diffeomorphism and it is said to be *hyperregular* if $\mathcal{L}$ is a global diffeomorphism. In the latter case $\mathcal{L}$ will be called Legendre transformation.

By (7.1) $\mathbf{L}$ is regular iff the matrix $(g^{\alpha\beta}_{i\,j}) := (\partial^\alpha_i \partial^\beta_j \mathbf{L})$ is nondegenerate in any system of coordinates, i.e. the second order differential of $\mathbf{L}|_{E_{\pi(u)}}$ is nondegenerate for every $u \in E$.

Let us put the relation between a Hamiltonian and a Lagrangian under the Legendre transformation $\mathcal{L}$:

$$\mathbf{H} = p^\beta_j y^j_\beta - \mathbf{L}. \tag{7.2}$$

We prove:

Proposition 7.1. *The inverse $\mathcal{L}^{-1}$ of the Legrende transformation $\mathcal{L}$ is*

(7.3) $$\mathcal{L}^{-1}(q^i, p_i^\alpha) = (x^i, y_\alpha^i = \partial_\alpha^i \mathbf{H}(x,p)).$$

Proof. We shall show that $\mathcal{L}^{-1} o \mathcal{L} = id|_E$ and conversely, $\mathcal{L} o \mathcal{L}^{-1} = id|_{E^*}$. Consider the Defintion 7.1. Since $\mathbf{L}(x,y)$ does not depend on $\partial_i^\alpha \mathbf{L}$ we get by direct calculation

(7.4) $$\begin{aligned}(x^i, y_\alpha^i) \xrightarrow{\mathcal{L}} (q^i, p_i^\alpha = \partial_i^\alpha \mathbf{L}) \xrightarrow{\mathcal{L}^{-1}} (x^i, y_\alpha^i = \partial_\alpha^i \mathbf{H}) = \\ = (x^i, \partial(y_\beta^j \mathbf{L} - \mathbf{L})/\partial(\partial_i^\alpha \mathbf{L})) = (x^i, \delta_j^i \delta_\alpha^\beta y_\beta^i - 0) = (x^i, y_\alpha^i).\end{aligned}$$

Conversely, since $\mathbf{L} = p_j^\beta y_\beta^j - \mathbf{H}$ and the function $\mathbf{H}$ does not depend on $\partial_\alpha^i \mathbf{H}$ we directly obtain

(7.5) $$\begin{aligned}(q^i, p_i^\alpha) \xrightarrow{\mathcal{L}^{-1}} (x^i, y_\alpha^i = \partial_\alpha^i \mathbf{H}) \xrightarrow{\mathcal{L}} (q^i, \partial \mathbf{L} | \partial(\partial_\alpha^i \mathbf{H})) = \\ = (q^i, \partial(p_j^\beta \partial_\beta^j \mathbf{H} - \mathbf{H})/\partial(\partial_\alpha^i \mathbf{H})) = (q^i, p_j^\beta \delta_i^j \delta_\beta^\alpha - 0) = (q^i, p_i^\alpha). \ \diamond\end{aligned}$$

If $\mathcal{L}^T$ and $(\mathcal{L}^{-1})^T$ are the *tangent maps* to $\mathcal{L}$ and $\mathcal{L}^{-1}$, then we have

(7.6) $$\begin{aligned}&(a) \quad \mathcal{L}^T(\partial_i) = \overset{*}{\partial}_i + \partial_i \partial_k^\beta(\mathbf{L}) \partial_\beta^k \\ &(b) \quad \mathcal{L}^T(\partial_i^\alpha) = g_{ij}^{\alpha\beta} \partial_\beta^j \\ &(c) \quad (\mathcal{L}^{-1})^T \overset{*}{\partial}_i = \partial_i + \overset{*}{\partial}_i \partial_\alpha^k(\mathbf{H}) \partial_k^\alpha \\ &(d) \quad (\mathcal{L}^{-1})^T(\partial_\alpha^i) = \partial_\alpha^i \partial_\beta^j(\mathbf{H}) \partial_j^\beta = g_{\alpha\beta}^{ij} \partial_j^\beta.\end{aligned} \qquad \begin{pmatrix} \overset{*}{\partial}_i := \partial | \partial q^i \\ \partial_i := \partial | \partial x^i \end{pmatrix}$$

By using these formulae we shall prove

Theorem 7.1. *If* $\mathbf{L}$ *is a hyperregular Lagrangian on* $E = \bigoplus_1^k TM$ *(i.e. the Legendre morphism associated to it is global diffeomorphism), then* $\mathcal{L}$ *carries a nonlinear connection* $\mathbf{N}$ *on* $E = \bigoplus_1^k TM$ *into a nonlinear connection* $\overset{*}{\mathbf{N}}$ *on* $\overset{*}{E} = \bigoplus_1^k T^*M$. *If* $N_{\alpha j}^i(x,y)$ *are the local coefficients of* $\mathbf{N}$ *and* $\overset{*}{N}{}_{ij}^\alpha(q,p)$ *are the local coefficients of* $\overset{*}{\mathbf{N}}$, *then we have*

(7.7) $$\overset{*}{N}{}_{ij}^\alpha(q,p) = -(\partial_\beta^k \overset{*}{\partial}_j(\mathbf{H}) + N_{\beta j}^k) g_{ki}^{\beta\alpha}.$$

Proof. From (7.6) (a) and (b) we deduce that

(7.8) $$\begin{aligned}\mathcal{L}^T(\delta_i) = \mathcal{L}^T(\partial_i - N_{\alpha i}^j \partial_j^\alpha) = \overset{*}{\partial}_i + \partial_i \partial_k^\beta(\mathbf{L}) \partial_\beta^k - N_{\alpha j}^j g_{jk}^{\alpha\beta} \partial_\beta^k = \\ = \overset{*}{\partial}_i + (\partial_i \partial_k^\beta(\mathbf{L}) - N_{\alpha i}^j g_{jk}^{\alpha\beta}) \partial_\beta^k.\end{aligned}$$

Moreover, from the defintion of **H** in (7.2) induced by **L** it follows

$$\text{(7.9)}\quad \begin{aligned}&\text{(a)}\quad \overset{*}{\partial}{}_i\mathbf{H} = -\partial_i\mathbf{L}\\ &\text{(b)}\quad \partial_i\partial_k^\beta(\mathbf{L}) = \partial_k^\beta(\partial_i\mathbf{L}) = -\partial_k^\beta(\overset{*}{\partial}{}_i\mathbf{H}) = -\partial_\alpha^j\partial_i(\mathbf{H})\partial_k^\beta p_j^\alpha = \\ &\qquad = -\partial_\alpha^j\partial_i(\mathbf{H})\partial_k^\beta\partial_j^\alpha\mathbf{L} = -\partial_\alpha^j\partial_i(\mathbf{H})g_{kj}^{\beta\alpha} \qquad (x^i = q^i).\end{aligned}$$

Hence for the local basis adapted to the horizontal distribution **N** on E ([8]) we get

$$\mathcal{L}^T(\delta_i) = \overset{*}{\partial}{}_i + (-\partial_\alpha^j\overset{*}{\partial}{}_i(\mathbf{H}) - N_{\alpha i}^j)g_{j\,k}^{\alpha\beta}\partial_\beta^k. \tag{7.10}$$

Putting

$$\overset{*}{N}{}_{ki}^{\beta} = -(\partial_\alpha^j\overset{*}{\partial}{}_i(\mathbf{H}) + N_{\alpha i}^j)g_{jk}^{\alpha\beta} \tag{7.11}$$

we have obtained that $\mathcal{L}^T$ maps $\{\delta_i\}$ to the local basis $\{\overset{*}{\delta}\}$ adapted to $\overset{*}{\mathbf{N}}$ on $\overset{*}{E}$ and the formula (7.7) holds. ◊

Remark 7.1. If **L** is only regular then Theorem 7.1. is valid only locally, i.e. on an open set of $\bigoplus_1^k TM$ for which $\mathcal{L}$ is diffeomorphism.

Remark 7.2 Even though $\mathcal{L}$ is only a local diffeomorphism the coefficients $\overset{*}{N}{}_{ki}^{\beta}$ define a global nonlinear connection since they satisfy the usual transformation law as it can be seen by a long calculation.

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