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THE ITERATES ARE NOT DENSE IN C

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Abstract: It is proved that the set of all iterates of continous functions are not dense in C.

Let C denote the set of continuous functions mapping $[0,1]$ into itself endowed with the sup norm. Denot by f^k the k^{th} iterate of the continuous function f. The structure of the set $W^k = \{f^k : f \in C\}$ was examined by M. Laczkovich and P.D. Humke. They proved in [1] and [2] that W^2 is not everywhere dense in C and W^k is an analytic non-Borel subset of C. The author of this paper proved in [3], [4] that the set $\bigcup\limits_{k>1} W^k$ of iterates of continuous functions is a first category set and W^2 is nowhere dense. The aim of this paper is to prove the following

Theorem. *The set* $\bigcup\limits_{k>1} W^k$ *of iterates of continuous functions is not everywhere dense in* C.

In other words: *there exists an open ball B* (see Figure 1) *such that B does not contain any iterates of any continuous function.*

The centre of the ball B is the continuous function g which is linear on $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$ and $g(0) = 0.03$, $g(\frac{1}{2}) = 0.99$, $g(1) = 0.03$ and the radius of the ball $r = 0.01$.

We introduce the following notations. We denote the lower boundary of B by $g_1(x) = g(x) - 0.01$ and the upper one by $g_2(x) = g(x) + 0.01$. Put $u_1 = \sup_{x \in [0,1]} g_1(x) = 0.98$, $u_2 = \inf \{g_1(x) | g_2(x) > u_1\} = u_1 - 2r = 0.96$, $u_3 = g_2(u_2)$ and $u_4 = g_2(u_3)$. $u_4 < g_2^{-1}(\frac{1}{2})$. Both g_1 and g_2 have only one fixed point, say r_1, r_2 respectively; put $D = [r_1, r_2]$. It is clear that for every $f \in B$ $\mathrm{Fix}(f) \subset D$ holds, where $\mathrm{Fix}(f) = \{x | f(x) = x\}$.

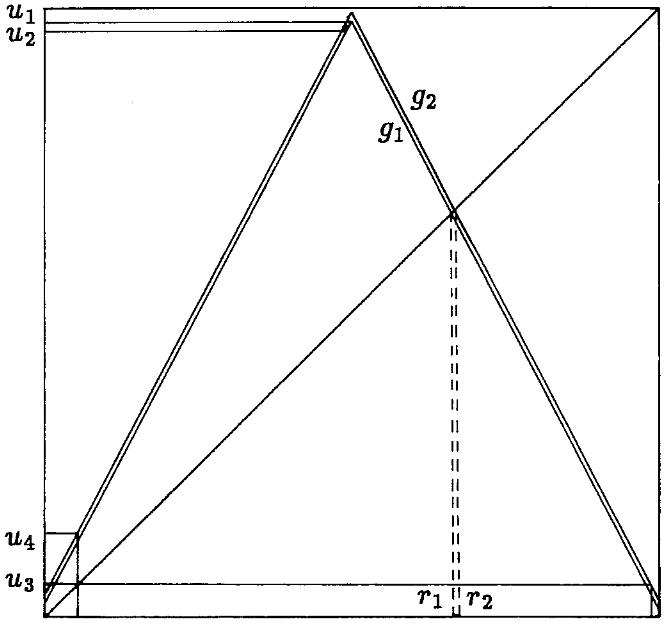

Figure 1

For every $H \subset [0,1]$ we denote by $\overline{H}$ the complement of H. Let $A, B \subset [0,1]$ we shall write $A < B$ if $a < b$ for every $a \in A$ and $b \in B$.

Proof of the Theorem. Assume that there exists $f \in B \cap \bigcup_{k>1} W^k$ say $f = \varphi^n$ for $\varphi \in C$ and $n > 1$. We define $I = \{x | g_2(x) > u_1\}$. We choose a, b such that $I = (a, b)$. It is easy to see that $u_4 < a$. For every $y \in \overline{I}$ $\varphi(y) \neq \varphi(\frac{1}{2})$ since $\varphi(y) = \varphi(\frac{1}{2})$ implies $f(y) = f(\frac{1}{2})$ which contradicts the definition of I.

There are four cases to consider:

Case 1. (i) $\varphi(\frac{1}{2}) < \varphi(\overline{I})$,
Case 2. (ii) $\varphi([0,a]) > \varphi(\frac{1}{2}) > \varphi([b,1])$,
Case 3. (iii) $\varphi([0,a]) < \varphi(\frac{1}{2}) < \varphi([b,1])$,
Case 4. (iv) $\varphi(\frac{1}{2}) > \varphi(\overline{I})$.

We prove that each of them leads to a contradiction.

Case 1: (i) holds. Now $\varphi(\frac{1}{2}) > a$ since otherwise $\mathrm{Fix}(\varphi) \cap [0,b] \neq \emptyset$ and thus $\mathrm{Fix}(f) \cap [0,b] \neq \emptyset$ which is impossible since $\mathrm{Fix}(f) \subset D$. Hence

$\varphi(\bar{I}) > \varphi(\frac{1}{2}) > a$ and, in particular, $\varphi(a) > a$. On the other hand, $f([0,1]) \cap [0,a) \neq \emptyset$ and hence there is y with $\varphi(y) < I$. Then $y \in I$ and $\varphi(y) < y$. It follows that $\mathrm{Fix}(\varphi) \cap I \neq \emptyset$. Therefore $\mathrm{Fix}(f) \cap I \neq \emptyset$ which is impossible since $\mathrm{Fix}(f) \subset D$ and $D \cap I = \emptyset$. $\diamondsuit$

Case 2: (ii) holds. First we prove

$$\varphi([0,u_3]) \cap [u_1,1] \neq \emptyset. \tag{1}$$

It is clear that (ii) implies

$$\min_{x\in[u_3,b]} \varphi(x) < \min_{x\in[0,u_3]} \varphi(x). \tag{2}$$

We also know that $\varphi([0,1]) \cap [u_1,1] \neq \emptyset$, hence by (ii), $\varphi([0,b]) \cap [u_1,1] \neq$ $\neq \emptyset$. Thus if (1) doesn't hold then

$$\max_{x\in[u_3,b]} \varphi(x) > \max_{x\in[0,u_3]} \varphi(x) \tag{3}$$

must hold. From (2) and (3) we get $\varphi([0,u_3]) \subset \varphi([u_3,b])$ and thus $f([0,u_3]) \subset f([u_3,b])$ which is false and (1) follows. Pick $x_0 \in$ $\in \varphi^{-1}([u_1,1]) \cap [0,u_3]$. Then

$$\varphi(f(x_0)) = f(\varphi(x_0)) < u_3 \tag{4}$$

holds by the definition of u_3. Since $f(x_0) < u_4$ (implied by $x_0 \in [0,u_3]$) from (4) we have

$$\min_{x\in[0,u_4]} \varphi(x) < u_3$$

and further (1) implies $\max_{x\in[0,u_4]} \varphi(x) > u_1$. Thus $\mathrm{Fix}(\varphi) \subset \varphi([0,u_4])$ since $\mathrm{Fix}(\varphi) \subset D \subset [u_3,u_1]$. Hence $f([0,u_4]) \cap \mathrm{Fix}(f) \neq \emptyset$ which contradicts $f \in B$. $\diamondsuit$

Case 3: (iii) holds. Let $d = \max \mathrm{Fix}(\varphi)$. First, we show

$$\varphi([0,d]) \leq u_1. \tag{5}$$

Suppose instead that

$$\exists m \leq d \text{ such that } \varphi(m) > u_1. \tag{6}$$

Since

$$\varphi(x) < x \text{ for every } x > d, \tag{7}$$

we have $\varphi([d,u_1]) < u_1$. From the assumptions (6) and (iii) it follows that $\varphi([a,m]) \supset \varphi([d,u_1])$ and thus $f([a,m]) \supset f([d,u_1])$ which contradicts $f \in B$. Thus (5) holds. Choose m such that $\varphi(m) > u_1$. From (5) and (7) we have $m > u_1$ whence $f(m) < u_4$. Thus $\exists 0 \leq j \leq n-1$ for which $[\varphi^j(m)\varphi^{j+1}(m)] \cap \mathrm{Fix}(\varphi) = \emptyset$. Let z be an arbitrary element of the set $[\varphi^j(m), \varphi^{j+1}(m)] \cap \mathrm{Fix}(\varphi)$. Then $z \in \varphi^j([\varphi(m), m])$ and hence

$$z = \varphi^{n-j}(z) \in f([\varphi(m), m]) \subset f([u_1, 1])$$

which is impossible as $z \in D$. $\Diamond$

Case 4: (iv) holds. Assume first that $n \geq 3$. We need 3 Lemmas.

Lemma 1. *Put $j = \min\{x \in [a,b] | \varphi(x) \geq u_1\}$. (It follows from* (iv) *that such a j exists.) Then the following inequalities hold:*

$$\varphi(f(j)) < u_3, \tag{8}$$

$$\varphi(f^2(j)) < u_4. \tag{9}$$

Proof. The relations $\varphi(j) \geq u_1$ and $f([u_1,1]) < u_3$ imply that $f(\varphi(j)) = \varphi(f(j)) < u_3$ which proves (8), while (9) follows from the definition of u_4:

$$\varphi(f^2(j)) = f^2(\varphi(j)) = f(f(\varphi(j))) \in f([0,u_3]) < u_4. \ \Diamond$$

Lemma 2. $\varphi([u_2,1]) < j$.

Proof. Assume that $\varphi([u_2,1]) < j$ doesn't hold. It follows from (8) that

$$\min_{x\in[u_2,1]} \varphi(x) < u_3 < j$$

thus $\exists x_0 \in [u_2,1]$ such that $\varphi(x_0) = j$. Hence

$$\varphi^2(x_0) \geq u_1. \tag{10}$$

On the other hand: $\varphi([0,u_3]) \cap \mathrm{Fix}(\varphi) = \emptyset$ since otherwise $f([0,u_3]) \cap \cap \mathrm{Fix}(f) \neq \emptyset$ would hold which is impossible. Thus from $f^2(j) \in [0,u_3]$ and from (9) we have $\min \varphi([0,u_3]) < \mathrm{Fix}(\varphi)$. Using (8) we find

$$\varphi^2(f(j)) = \varphi(\varphi(f(j))) \in \varphi([0,u_3]) < \mathrm{Fix}(\varphi).$$

From this and (10) we get

$$\min_{x\in[u_2,1]} \varphi^2(x) < \mathrm{Fix}(\varphi) < u_1 \leq \max_{x\in[u_2,1]} \varphi^2(x),$$

thus $\varphi^2([u_2,1]) \supset \mathrm{Fix}(\varphi)$, that is $f([u_2,1]) \cap \mathrm{Fix}(f) \neq \emptyset$ contradicting $f \in B$. $\diamondsuit$

Since $f(j) > u_1$ and $f^2(j) < u_3$, it follows from (9) that $\varphi([0,u_3])\cap$ $\cap[0,u_4] \neq \emptyset$. On the other hand, $\varphi(j) \geq u_1 > j$ and, as $u_3 < u_4 < j$, it follows that there is a $v_2 < j$ such that $\varphi(v_2) = j$.

Lemma 3. $\varphi^{-1}(v_2) \cap (0,j) \neq \emptyset$.

Proof. We first show that $u_4 < v_2$. Assume that $v_2 \leq u_4$. Then

$$\varphi([0,u_4]) \supset [u_4,j] \tag{11}$$

since $\varphi(v_2) = j$ and $\varphi(f^2(j)) < u_4$. Further $\varphi([0,u_4]) < \mathrm{Fix}(\varphi)$, since otherwise $\mathrm{Fix}(f) \cap f([0,u_4]) \neq \emptyset$ which is impossible. Now by (11) $\varphi^2([0,u_4]) \supset \varphi([u_4,j]) \supset [\varphi(u_4),\varphi(j)] \supset \mathrm{Fix}(\varphi)$ but this implies

$$f([0,u_4]) \cap \mathrm{Fix}(f) \neq \emptyset,$$

a contradiction. Thus $u_4 < v_2$. We know that $\varphi(f^2(j)) \in [0,u_4]$ so using (8) and the definition of v_2 we get a point z such that $\min f^2(j) < z < v_2$ and $\varphi(z) = v_2$. Thus $\varphi^{-1}(v_2) \cap (0,j) \neq \emptyset$. $\diamondsuit$

Choose $v_1 \in \varphi^{-1}(v_2) \cap (0,j)$; the action of the first 3 iterates on v_1 is shown bellow:

$$v_1 \overset{\varphi}{\Rightarrow} v_2 \overset{\varphi}{\Rightarrow} j \overset{\varphi}{\Rightarrow} \varphi(j) \in [u_1,1].$$

Then $\varphi^3(v_1) \geq u_1$ and by Lemma 2 $\varphi^3(\varphi_2) = \varphi(v^3(\varphi_1)) < j$. Thus we get

$$\varphi^3([v_1,v_2]) \supset [j,u_1]. \tag{12}$$

But $\varphi(j) \geq u_1$ and it follows from Lemma 2 that $\varphi(u_1) < j$ whence

$$\varphi([j,u_1]) \supset [j,u_1]. \tag{13}$$

From (12) and (13) we get

$$\begin{aligned} f([v_1,v_2]) = \varphi^{n-3}(\varphi^3[v_1,v_2]) \supset \varphi^{n-3}([j,u_1]) \supset \\ \supset \varphi^{n-4}([j,u_1]) \supset \ldots \supset [j,u_1] \end{aligned}$$

and it follows from the definition of I that $[v_1,v_2] \cap I \neq \emptyset$, whence $v_2 \in I$. Thus

$$\varphi([v_2,j]) \supset [\varphi(v_2),\varphi(j)] \supset \mathrm{Fix}(\varphi)$$

and further $[v_2,j] \supset I$ since $v_2 \in I$. Thus we get $f(I) \cap \mathrm{Fix}(f) \neq \emptyset$ which is a contradiction.

It remains to consider Case 4 with $n = 2$. We keep the definition of j from Lemma 1. We have

$$\varphi(j) \in [u_2, 1] \quad \text{and} \quad \varphi(\varphi(j)) \in [u_2, 1]. \tag{14}$$

Whence $\varphi([u_2,1]) \cap [u_2,1] \neq \emptyset$ but $\varphi(f(j)) = f(\varphi(j)) < u_3$ shows that $\varphi([u_2,1]) \cap [0,u_3] \neq \emptyset$ and hence $f([u_2,1]) \cap \text{Fix}(f) \neq \emptyset$ which is a contradiction. $\diamondsuit$

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