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BERNSTEIN APPROXIMATION OF A FUNCTION WHICH DERIVATIVES SATISFY THE LIPSCHITZ CONDITION ON BOUNDED SQUARE OR TRIANGLE*

Jerzy **Albrycht**

Institute of Mathematics, Technical University of Poznań, ul. Piotrowo 3a, PL-60-965 Poznań, Poland.

Adam **Marlewski**

Institute of Mathematics, Technical University of Poznań, ul. Piotrowo 3a, PL-60-965 Poznań, Poland.

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Abstract: Let Bf be the Bernstein polynomial of a function f such that its first derivatives satisfy the Lipschitz condition of order 1 on the unit square or on the standard triangle. It is shown that the approximation-error of the function f by the polynomial Bf does not exceed a quantity depending on the Lipschitz constants and the degree of Bf only. This way it is the full analogue to the one-dimensional case observed first by A.O. Geldfond.

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1. One-dimensional case. Gelfond has shown [1] that

$$\max_{0\leq x\leq 1}|B_nf(x)-f(x)|\leq \tfrac{1}{4n}L$$

for every function $f\in C^1_{<0,1>}$ such that its derivative f' satisfies the Lipschitz condition of degree 1 and constant L. Here B_nf stands for the classical Bernstein polynomial of degree n built for a function f, i.e. $B_nf(x):=\Sigma_{j=0}^{n}\left(\frac{j}{n}\right)p_{n,j}(x)$, where $p_{n,j}(x):=\binom{n}{j}x^j(1-x)^{n-j}$ and $x\in<0,1>$.

2. Approximation on the unit square. In 1933 Hildebrandt and Schoenberg have extended the notion of Bernstein polynomial to the case when a function f being approximated is defined on the unit square

$$K:=\{(x,y)\in\mathbb{R}^2:0\leq x,y\leq 1\}.$$

They set

$$B_{m,n}f(x,y):=\sum_{j=0}^{m}\sum_{k=0}^{n}f\left(\frac{j}{m},\frac{k}{n}\right)p_{m,j}(x)p_{n,k}(y)$$

and they proved that $B_{m,n}f$ tends to the function f uniformly in the space C_K (of all functions continuous on the domain K) as m and n increase to the infinity.

The polynomials $B_{m,n}f$ have been investigated a.o. by Butzer and Aramă (for details on bibliography see [2]). In the fifties they have proved the analogues of the classical one-dimensional cases (concerning the approximations of the derivatives of the function f by the derivatives of the polynomial $B_{m,n}f$ and the preserving of the convexity of the function f by its polynomial $B_{m,n}f$, respectively). Here we complete these analogues, namely we give the analogue to the Gelfond's result listed in Part 1, that is we show that there holds the following

Theorem 1. *Let f_1 and f_2 be the first derivatives* (with respect to the first and the second argument, respectively) *of a function $f\in C^1_K$ and let L_1,L_2 be the positive constants such that*

$$(*)\qquad |f_j(x,y)-f(s,t)|\leq L_1|x-s|+L_2|y-t|$$

for $j=1,2$ and for every points $(x,y),(s,t)\in K$ (we assume here that $|x-s|$, $|y-t|\leq 1$, naturally). *Then*

$$|B_{m,n}f(x,y)-f(x,y)|\leq \tfrac{1}{4}\left(\tfrac{1}{m}L_1+\tfrac{1}{n}L_2\right).$$

Proof. We will use the identities

$$\sum_{j=0}^{m}\left(\frac{j}{m}\right)^r p_{m,j}(x) = x^r \qquad \text{for} \qquad r = 0,1$$

and the following form of the Mean Value Theorem

$$f(x,y) - f(s,t) = f_1(\sigma,y)(x-s) + f_2(x,\tau)(y-t),$$

where σ and τ are the points laying somewhere in the intervals $(x-|x-s|,\, x+|x-s|)$ and $(y-|y-t|,\, y+|y-t|)$, respectively. Applying the above and the Lipschitz condition $(*)$ we obtain

$$|f(x,y) - f(s,t) - f_1(s,t)(x-s) - f_2(s,t)(y-t)| \leq$$
$$\leq L_1(x-s)^2 + L_2(y-t)^2.$$

Therefore the remainder $R := B_{m,n}f(x,y) - f(x,y)$ can be estimated as follows

$$|R| \leq \sum_{j=0}^{m}\sum_{k=0}^{n}\{|f_1(\sigma,\frac{k}{n}) - f_1(x,y)| \cdot |x - \frac{j}{m}| +$$
$$+ |f_2(\frac{j}{m},\tau) - f_2(x,y)| \cdot |y - \frac{k}{n}|\} p_{m,j}(x)p_{n,k}(y) \leq$$
$$\leq L_1\frac{x(1-x)}{m} + L_2\frac{y(1-y)}{n} \leq \frac{1}{4}\left(\frac{1}{m}L_1 + \frac{1}{n}L_2\right). \ \Diamond$$

3. Approximation on the standard simplex. Now we investigate the case when a function f is defined on the standard simple

$$T := \{(x,y) \in \mathbb{R}^2 : 0 \leq x, y, x+y \leq 1\}.$$

On this triangle there are defined the Steffensen polynomials $p_{m,j,k}$

$$p_{m,j,k}(x,y) := \binom{m}{j}\binom{m-j}{k} x^j y^k (1-x-y)^{m-j-k}.$$

Making use of them one can define (see [3]) the following polynomials $S_m f$,

$$S_m f(x,y) := \sum_{j=0}^{m} \sum_{k=0}^{m-j} f(\tfrac{j}{m}, \tfrac{k}{m}) p_{m,j,k}(x,y).$$

These polynomials, called the Bernstein polynomials on the triangle T, were investigated a.o. by Stancu (1960) and Lupaş (1974) whose obtained the analogical theorems to **Butzer's and Aramă's** results. That analogues can be completed (comp. **Theorem** 1) by the following

Theorem 2. *If $f \in C_T^1$ and if the Lipschitz condition* $(*)$ *holds true for $j = 1, 2$ on the whole triangle T, then*

$$|S_m f(x,y) - f(x,y)| \leq \tfrac{1}{4m}(L_1 + L_2).$$

Proof goes similarly to the proof of Theorem 1, one has only use the identities

$$\sum_{j=0}^{m} \sum_{k=0}^{m-j} p_{m,j,k}(x,y) = 1,$$

$$\sum_{j=0}^{m} \sum_{k=0}^{m-j} \tfrac{j}{m} p_{m,j,k}(x,y) = x,$$

$$\sum_{j=0}^{m} \sum_{k=0}^{m-j} \tfrac{k}{m} p_{m,j,k}(x,y) = y. \ \diamondsuit$$

4. Multidimensional cases. Using the same technique as in Part 2 and Part 3 one can easily obtain the *multidimensional analogues* of the Theorems 1 and 2 concerning the Bernstein approximation on the cubes

$$\{(x_1, x_2, \ldots, x_d) : 0 \leq x_j \leq 1 \quad \text{for} \quad j = 1, 2, \ldots, d\}$$

and on the simplexes

$$\{(x_1, x_2, \ldots, x_d) : 0 \leq x_j \leq 1 \quad \text{for} \quad j = 1, 2, \ldots, d \quad \text{and} \ \sum_{j=1}^{d} x_j \leq 1\}.$$

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