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CENTRALIZER NEAR-RINGS ACTING ON SE-GROUPS

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Abstract: Let $N = C(A, G)$ be a centralizer near-ring determined by a group A of automorphisms of the group G such that the identity of N is the sum of a finite number of mutually orthogonal primitive idempotents, e_i. A group M is called an SE-group for N if N acts as a semigroup of endomorphisms on M with an aditional "strong" property for the idempotents, e_i. In this paper we investigate the structure of the centralizer near-ring $C(N, M)$ and as an application obtain a near-ring analogue to a well known matrix theory result.

1. Introduction

Let R be a ring with 1. Then R forms a left unital R-module ${}_RR$ and a right unital R-module R_R. With each of these R-modules we have the ring $\text{End}_R({}_RR)$ of R-endomorphisms of ${}_RR$ and the ring $\text{End}_R(R_R)$ of R-endomorphisms of R_R. It is easy to see that $\text{End}_R({}_RR)$ is anti-isomorphic to R and $\text{End}_R(R_R)$ is isomorphic to R.

This left-right duality for rings fails for near-rings. Let N be a (right) zero-symmetric near-ring with 1 which does not satisfy the left distributive law. Then ${}_NN$ is a left unital N-module (see Pilz [6] or Meldrum [4] for near-ring terminology and basic facts). However, since N does not satisfy the left distributive law, there exist elements n_1, n_2 and n_3 in N such that $n_1(n_2 + n_3) \neq n_1n_2 + n_1n_3$, violating a right N-module axiom.

The near-ring analogy to $\text{End}_R({}_RR)$ is the set $\text{Map}_N({}_NN)$ consisting of all maps f from N into N such that $f(nm) = nf(m)$ for all n in N and all m in ${}_NN$. As in the ring case $\text{Map}_N({}_NN)$ consists precisely of right multiplication maps by elements of N, but if N does not satisfy the left distributive law then, under function addition and function composition, $\text{Map}_N({}_NN)$ does not form a near-ring since it is not closed under addition.

Using N_N we see that $\text{Map}_N(N_N)$ consists of left multipliation maps by elements of N and it forms a (right) near-ring under function addition and function composition. The near-ring N acts on N_N as a semigroup (under function composition) of endomorphisms of the group $(N_N, +)$. For if n_1, n_2 are in N_N and n is in N then $(n_1 + n_2)n = = n_1n + n_2n$. Since the left multiplication maps $\text{Map}_N(N_N)$ are precisely the functions on N that commute with the right multiplication maps so $\text{Map}_N(N_N)$ is the centralizer near-ring $C(N, N_N)$ where N acts on N_N as a semigroup of endomorphisms via right multiplication. (See [2] for details about centralizer near-rings $C(S, G)$ where S is a semigroup of endomorphisms of the group G.)

The prototype of a finite-ring is the centralizer near-ring $C(A, G)$ where G is a finite group and A is a group of automorphisms of G (see [2]). If $N = C(A, G)$ then the identity 1 of N is the sum of mutually orthogonal primitive idempotents, $1 = e_1 + e_2 + \cdots + e_t$. Moreover we have, for every i, j with $i \neq j, n(e_i + e_j) = ne_i + ne_j$ for all n in N, and $e_i + e_j = e_j + e_i$. This implies that if n belongs to N such that $ne_i = 0$ for every i then $n = 0$. With this in mind we have the following definition where $N = C(A, G)$.

Definition. If $N = C(A, G)$ with $1 = e_1 + e_2 + \cdots + e_t$ as above, then a group $(M, +)$ is an *SE-group (strong endomorphism group)* for N if

there is a composition $M \times N$ to M such that
(a) $(m_1 + m_2)n = m_1 n + m_2 n$ for every m_1, m_2 in M and n in N,
(b) $(mn_1)n_2 = m(n_1 n_2)$ for every m in M and n_1, n_2 in N,
(c) $m1 = m$ for every m in M,
(d) $m0 = 0$ for every m in M, and
(e) if m in M is sucht that $me_i = 0$ for all i, then $m = 0$.

Thus, when M is an SE-group for $N = C(A, G)$, the first four axioms require that N acts as a semigroup of endomorphisms on M with 1 acting as the identity map and 0 as the zero map, while the "strong" property (e) leads to $m(e_i + e_j) = me_i + me_j$ for all i, j with $i \neq j$.

We note that if M_1 and M_2 are SE-groups for N then so is $M_1 + M_2$. Also if R is a right ideal of N then R is an SE-group for N. In particular, N_N is an SE-group for N. Moreover, with each SE-group M for N we have the corresponding centralizer near-ring $C(N, M)$.

It is the purpose of this article to investigate the structure of the near-ring $C(N, M)$ where N is a finite centralizer near-ring of the type $C(A, G), A$ is a group of automorphisms of G and M is an SE-group for N. In the next section we focus on the case where $N = C(A, G)$ is a simple near-ring. In section 3 we present two general results and in the final section we use a theorem of A.P.J. van der Walt ([8]) to obtain a near-ring analogue of a well-known matrix theory result.

2. Structure of $C(N, M), N$ simple

In this section N represents a finite centralizer near-ring $C(A, G)$ where A is a group of automorphisms of the finite group G and $1 = e_1 + e_2 + \cdots + e_t$ where the e_i's are mutually orthogonal primitive idempotents. We recall that if N is simple then there exists a group G and a fixed point free group A of automorphisms of G such that N is isomorphic to $C(A, G)$.

Lemma 1. *Let $N = C(A, G)$ with $1 = e_1 + e_2 + \cdots + e_t$ and let M be an SE-group for N. If f belongs to $C(N, M)$ then*
(a) *$f(Me_i)$ is a subset of Me_i for every i and*
(b) *$f(m_1 e_1 + m_2 e_2 + \cdots + m_t e_t) = f(m_1 e_1) + f(m_2 e_2) + \cdots + f(m_t e_t)$ for all m_i in M.*

Proof. (a) For m in $M, f(me_i) = f(me_ie_i) = f(me_i)e_i$ which is in Me_i.

(b) $f(m_1e_1+m_2e_2+\cdots+m_te_t) = f(m_1e_1+m_2e_2+\cdots+m_te_t)(e_1+ +e_2+\cdots+e_t) = f(m_1e_1+m_2e_2+\cdots+m_te_t)e_1+f(m_1e_1+m_2e_2+\cdots+ +m_te_t)e_2+\cdots+f(m_1e_1+m_2e_2+\cdots+m_te_t)e_t = f(me_1)+f(me_2)+ +\cdots+f(me_t)$. ◇

Lemma 2. *If N is simple with $N = C(A,G)$ where A is fixed point free then f in $C(N,M)$ is completely determined by its action on the set Me_1.*

Proof. For $i \neq 1$ there exist elements e_{i1} and e_{1i} in N such that $e_{i1}e_{1i} = e_i$, $e_{1i}e_{i1} = e_1$, $e_{i1}e_1 = e_{i1}$ and $e_{1i}e_i = e_{1i}$. We have $Me_i = = Me_{i1}e_{1i}$ and so $f(me_i) = f(me_{i1}e_{1i}) = f(me_{i1})e_{1i}$. Since me_{i1} belongs to Me_1 so $f(me_{i1})$ is known and f is determined on Me_i. Since M is a sum of the Me_i's, f is determined on M by Lemma 1, part (b). We note that the extension of f is unique, for if $f(Me_1) = \{0\}$ then $f(me_i) = f(me_{i1})e_{1i} = 0$ and so f is the zero map. ◇

Our first theorem characterizes $C(N,M)$ when N is simple.

Theorem 1. *Let $N = C(A,G)$ be a finite simple near-ring where A is a fixed point free group of automorphisms of G. Let M be an SE-group for N. Then $C(N,M)$ is isomorphic to $C(N_{11}^*, Me_1)$ where N_{11}^* is the set of nonzero elements in e_1Ne_1 and acts on Me_1 by right multiplication.*

Proof. Define ψ from $C(N,M)$ to $C(N_{11}^*, Me_1)$ by $\psi(f) = f$ restricted to the set Me_1. By Lemma 1, $\psi(f)$ is a function on the group Me_1. Since f belongs to $C(N,M)$, $f(me_1)n_{11} = f(me_1n_{11})$ where n_{11} is in N_{11}^*. This means $\psi(f)$ belongs to $C(N_{11}^*, Me_1)$. The function ψ is one-to-one by Lemma 3. That ψ preserves sums and products in $C(N,M)$ is easily checked.

It remains to show that ψ is onto. To this end, select g in $C(N_{11}^*, Me_1)$. The function g is already defined on Me_1 and we need to extend g to all of M. Define g on Me_i as follows: $g(me_i) = g(me_{i1})e_{1i}$. We show that g is well defined. For suppose $m_1e_i = m_2e_i, m_1, m_2$ in M. Then $(m_1e_{i1} - m_2e_{i1})e_{1i} = 0$. Hence $(m_1e_{i1} - m_2e_{i1})e_1 = 0$ and since $(m_1e_{i1} - m_2e_{i1})e_j = 0$ for $j = 2, \ldots, t$, we have $m_1e_{i1} - m_2e_{i1} = 0$ by property (e) of the definitionn of SE-group. Extend g to all of M additively, that is $g(m) = g(me_1+me_2+\cdots+me_t) = g(me_1)+ +g(me_2)+\cdots+g(me_t)$. It remains to show that the extended function g belongs to $C(N,M)$, i.e. that $g(mn) = g(m)n$ for every m in M and

n in N. We have $g(mn) = g(mne_1) + g(mne_2) + \cdots + g(mne_t)$ and $g(m)n = g(m)ne_1 + g(m)ne_2 + \cdots + g(m)ne_t$, so it suffices to show that $g(mne_i) = g(m)ne_i$ for each i. Since N is a centralizer near-ring, $ne_i = e_jne_i$ for some index j (which depends on n). So we have

$$\begin{aligned}
g(mne_i) &= g(me_jne_i) \\
&= g(me_{j1}e_{1j}ne_{i1}e_{1i}) \\
&= g(me_{j1}(e_{1j}ne_{i1}))e_{1i} \qquad \text{(definition of } g) \\
&= g(me_{j1}(e_{1j}ne_{i1})(ne_{1i} \qquad (g \text{ belongs to } C(N_{11}^*, Me_1)) \\
&= g(me_{j1})(e_{1j}ne_i) \\
&= g(me_{j1}e_{1j})ne_i \qquad \text{(definition of } g \text{ on } M_{e_j}) \\
&= g(me_j)ne_i \\
&= [g(me_1) + g(me_2) + \cdots + g(me_t)]ne_i \ (\text{since } ne_i = e_jne_i \text{ and} \\
&\quad g(me_j) \text{ belongs to } Me_j \text{ for each } j) \\
&= g(m)ne_i. \ \diamondsuit
\end{aligned}$$

We remark that since centralizer near-rings are zero-symmetric, $C(N_{11}^*, Me_1) = C(e_1Ne_1, Me_1)$. In the sequel we will often use this observation.

By specialzing M we obtain several applications of Theorem 1. Our first application is obtained by letting $M = N_N$.

Lemma 3. *Let N be the finite simple near-ring $C(A, G)$ where A is a fixed point free group of automorphisms of G and $1 = e_1 + e_2 + \cdots + e_t$, mutually orthogonal primitive idempotents. Then*

(a) *N_{11}^* is a multiplicative group anti-isomorphic to A,*

(b) *Ne_1 is an additive group isomorphic to G,*

(c) *N_{11}^* acts on Ne_1 by right multiplication as a fixed point free group of automorphisms, and*

(d) *$C(N_{11}^*, Ne_1)$ is isomorphic to $C(A, G)$.*

Proof. (a) Let v_1 be a nonzero element in G sucht that $e_1(v_1) = v_1$. Then v_1 belongs to Av_1 and $N_{11}^* = \{f \text{ in } N | f(v_1) \text{ belongs to } Av_1 \text{ and } f(w) = 0 \text{ for all } w \text{ not in } Av_1\}$. Since A is fixed point free it follows that for every σ in A there exists a unique f in N_{11}^* such that $f(v_1) = \sigma v_1$. Define ψ from N_{11}^* to A by $\psi(f) = \sigma$ where $f(v_1) = \sigma v_1$. If f, g are in N_{11}^* with $f(v_1) = \sigma v_1$ and $g(v_1) = \sigma' v_1$ then $(gf)(v_1) = g(\sigma v_1) = \sigma g(v_1) = \sigma\sigma' v_1$. So $\psi(gf) = \sigma\sigma' = \psi(f)\psi(g)$. The function ψ is clearly one-to-one and onto.

(b) Since A is fixed point free, if $\sigma \neq 1$ belongs to A and if v is nonzero in G then $\sigma(v) \neq v$. This implies (see [2]) that if v_1 is as in part (a) then for every w in G there exists a function f in N such that $f(v_1) = w$ and $f(v) = 0$ for all v not Av_1. Moreover f is unique with this property and f belongs to Ne_1. Define φ from Ne_1 to G by $\varphi(f) = w$. Now φ is easily seen to be an isomorphism of Ne_1 onto G.

(c) If n_{11} belongs to N_{11}^* then define the map $R_{n_{11}}$ from Ne_1 to Ne_1 by $R_{n_{11}}(ne_1) = ne_1n_{11}$. The map $R_{n_{11}}$ is clearly an endomorphism of the group $(Ne_1, +)$. Moreover it is an automorphism, for if $R_{n_{11}}(ne_1 = 0$ then $ne_1n_{11} = 0$. But N_{11}^* is a group with identity e_1 under multiplication so n_{11} has an inverse n_{11}^{-1} and $0 = 0n_{11}^{-1} = = (ne_1n_{11})n_{11}^{-1} = ne_1$, which implies $R_{n_{11}}$ is one-to-one. Since Ne_1 is finite the map is onto.

To show N_{11}^* acts fixed point freely on Ne_1 suppose $R_{n_{11}}(ne_1) = = ne_1 \neq 0$. Then $ne_1 = ne_1n_{11}$. We have $ne_1 = e_jne_1$ for some j. Since N is simple there exists m_{ij} in e_1Ne_j such that $m_{1j}ne_1 = e_1$. This means $e_1 = m_{1j}ne_1 = m_{1j}ne_1n_{11}$, and $R_{n_{11}} = R_{e_1}$ which is the identity map on N_{11}.

The correspondence $R_{n_{11}}$ to n_{11} is an anti-isomorphic of $\{R_{n_{11}} | n_{11}$ belongs to $N_{11}^*\}$ with N_{11}^* and since the latter is anti-isomorphic to A, $\{R_{n_{11}} | n_{11}$ belongs to $N_{11}^*\}$ is isomorphic to A.

(d) To show $C(N_{11}^*, Ne_1)$ and $C(A, G)$ are isomorphic it suffices to show that the pair (N_{11}^*, Ne_1) is isomorphic to the pair (A, G) by way of a semi-linear transformation ψ from Ne_1 onto G (see Maxson and Smith [3] or Ramakotaiah [7]). (Here N_{11}^* is identified with the right multiplication maps by elements of the set N_{11}^*.) Let ψ be the isomorphism from Ne_1 to G defined as in (b) and let β from $\{R_{n_{11}} | n_{11}$ belongs to $N_{11}^*\}$ to A be the isomorphism as developed in (c). Then $\psi(R_{n_{11}}f) = \psi(fn_{11}) = fn_{11}(v_1) = f(n_{11}v_1) = f(\beta v_1) = \beta f(v_1) = = R_{n_{11}}^{\beta}\psi(f)$, and ψ is our desired one-to-one semi-linear transformation. $\diamondsuit$

This leads to the following application of Theorem 1.

Corollary 1. *Let N be a finite simple near-ring with $N = C(A, G)$ where A is fixed point free. Then $C(N, N_N)$ is isomorphic to N.*

Proof. By Theorem 1, $C(N, N_N)$ is isomorphic to $C(N_{11}^*, Ne_1)$ which is isomorphic to N by Lemma 3. $\diamondsuit$

Corollary 2. *Let N be a finite simple near-ring with $N = C(A, G)$ where A is fixed point free. If k is a positive integer let $N^k =$*

$= N \oplus N \oplus \cdots \oplus N$ *(k direct summands). Then* $C(N, N^k)$ *is isomorphic to* $C(A, G^k)$*. In particular* $C(N, N^k)$ *is simple.*

Proof. By Theorem 1, $C(N, N^k)$ is isomorphic to $C(N_{11}^*, N^k e_1)$. As in the proof of Lemma 3, $N^k e_1$ is isomorphic to G^k and N_{11}^* acts on $N^k e_1$ fixed point freely by right multiplication. Also as in the proof of Lemma 1 the pairs $(N_{11}^*, N^k e_1)$ and (A, G^k) are isomorphic via a semi-linear transformation. So $C(N, N^k)$ is isomorphic to $C(A, G^k)$ and since A acts fixed point freely on G^k, $C(N, N^k)$ is simple. $\diamondsuit$

If $N = C(A, G)$ and if R is a right ideal of N then $M = R$ is an SE-group for N. Our next application of Theorem 1 deals with this situation. First we describe the right ideals in the simple near-ring $N = C(A, G)$.

Lemma 4. *Let* N *be a finite simple near-ring with* $N = C(A, G)$ *where* A *is fixed point free. A nonempty subset* R *of* N *is a right ideal of* N *if and only if there exists an* A*-invariant subgroup* H *of* G *such that* $R = e_H N$ *where* e_H *in* N *is the idempotent map on* G *which is the identity on* H *and zero off* H*.*

Proof. If H is an A-invariant subgroup of G and if $R = e_H N$ it is easily verified that R is a right ideal on N.

Now assume R is a right ideal of N. Let $H = \{w \text{ in } G \mid \text{there is a } v \text{ in } G \text{ and an } f \text{ in } R \text{ with } f(v) = w\}$. To show H is an A-invariant subgroup of G select $w \neq 0$ in H. Then there exists a $v \neq 0$ in G and an f in R such that $f(v) = w$. For β in A we have $f(\beta v) = \beta f(v) = \beta w$, an element of H. So H is A-invariant. Since N is simple and A is fixed point free, it follows that for every $v \neq 0$ in G and any u in G there exists an n in N such that $n(v) = u$. So if w belongs to H with $g(u) = w$ where g is in R then $g(u) = gn(v) = w$ and gn belongs to R since R is a right ideal. This means $H = Rv$ for every nonzero v in G. But Rv is clearly a group, so H is a subgroup of G.

Let e_H be the idempotent in N which is the identity on H and 0 off H. We show now that e_H belongs to R. For $h \neq 0$ in H let e_h be the idempotent in N which is the identity on Ah and 0 elsewhere. Since $Rh = h$ and $e_h(h) = h$ we have $Re_h h = H$. The elements (maps) of Re_h are all 0 off Ah, so there exists an re_h in Re_h such that $re_h(h) = h$ and re_h is 0 off Ah. This means $re_h = e_h$ and e_h belongs to R. Since e_h belongs to R for all nonzero h in H and since H is finite, e_H belongs to R (e_H is the sum of e_h's, one h for each nonzero A-orbit in H). $\diamondsuit$

Corollary 3. *Let* N *be a finite simple near-ring with* $N = C(A, G)$

where A is a fixed point free group of automorphisms. Let $R = e_H N$ be a right ideal of N. Then $C(N, R)$ is isomorphic to $C(N_{11}^*, Re_1)$ which in turn is isomorphic to $C(A, h)$, a simple near-ring.

Proof. That $C(N, R)$ is isomorphic to $C(N_{11}^*, Re_1)$ is clear from Theorem 1. To see that N_{11}^* acts fixed point freely on the group Re_1 by right multiplication it is enough to use Lemma 3, part (c) since Re_1 is a subset of Ne_1.

We have Re_1 isomorphic to H and as in the proof of Lemma 3, part (d) the pairs (N_{11}^*, Re_1) and (A, H) are isomorphic via a semi-linear transformation. So the simple near-ring $C(N_{11}^*, Re_1)$ is isomorphic to $C(A, H)$. $\diamondsuit$

Corollary 4. *Let N be a finite simple near-ring with $N = C(A, G)$ where A is fixed point free. If $R = e_H N$ is a right ideal of N then for any positive integer $k, C(N, R^k)$ is isomorphic to $C(N_{11}^*, (Re_1)^k)$ which in turn is isomorphic to $C(A, H^k)$, a simple near-ring.*

Proof. Similarly to that of Corollary 2. $\diamondsuit$

Corollary 5. *Let N be a finite simple near-ring with $N = C(A, G)$ where A is fixed point free. If $R = e_H N$ is a right ideal of N such that $(R, +)$ is a normal subgroup of $(N, +)$, then N acts on N/R by $(a + R)n = an + R$ and $C(N, N/R)$ is isomorphic to $C(N_{11}^*, (N/R)e_1)$ which in turn is isomorphic to the near-ring $C(A, G/H)$. Moreover $C(A, G/H)$ is simple.*

Proof. The first isomorphism is from Theorem 1. Since $(R, +)$ is normal in $(N, +)$ so H is normal in G. One checks that $(N_{11}^*, (N/R)e_1)$ and $(A, G/H)$ are isomorphic via a semi-linear transformation.

To see that $C(A, G/H)$ is simple it suffices to see that A acts fixed point freely on G/H. Suppose $\beta \neq 1$ belongs to A and that $\beta(v + H) = v + H$. This means $-v + \beta v$ belongs to H. We recall (see [1]) that a fixed point free automorphism β on a finite group G has the property that every x in G has the unique form $-x + \beta x$. Since β acts fixed point freely on H and since $-v + \beta v$ belongs to H we have $-v + \beta v = -w + \beta w$ for some w in H. This implies $v = w$ and $v + H = H$, i.e. $v + H$ is the identity element of G/H. $\diamondsuit$

Corollary 6. *Let N be a finite simple near-ring with $N = C(A, G)$ where A is fixed point free. If $R = e_H N$ is a right ideal of N such that $(R, +)$ is a normal subgroup of $(N, +)$ and if k is a positive integer then $C(N, (N/R)^k)$ is isomorphic to $C(N_{11}^*, (N/R)^k)$ which in turn is isomorphic to $C(A, (G/H)^k)$.*

Proof. Similar to that of Corollary 2. ◇

If N is simple and if M is an SE-group then $C(N, M)$ need not be simple as the following example shows.

Example. Let N be $GF(4)$, the finite field with 4 elements. The field N is clearly a simple near-ring and $N = C(A, G)$ where G is the group $(GF(4), +)$ and A is the fixed point free automorphism group on $GF(4)$ consisting of the right multiplication maps by the three nonzero elements of $GF(4)$.

Let $N = \{0, 1, a, a^2\}$, then $A = \{1, R_a, R_{a^2}\}$. Let $M = S_3$, the symmetric group on three elements. Define the action of N on M as follows: if β is in S_3 then

$$\begin{aligned} \beta 0 &= (1) \\ \beta 1 &= \beta \\ \beta a &= (123)^{-1}\beta(123) \\ \beta a^2 &= (132)^{-1}\beta(132). \end{aligned}$$

So right multiplication by 0 is the zero endomorphism of S_3, by 1 is the identity map, by a is the automorphism which is conjugation by (123), and by a^2 is the automorphism which is conjugation by (132). With this action of N on M, M forms an SE-group for N. But $C(N, M) = C(N^*, S_3)$ is not simple since (123) in $M = S_3$ is fixed by all the nonzero elements N^* in N and (12) is not (so there is stabilizer containment, see [2]).

Let N be a finite near-ring with $N = C(A, G)$ where A is a fixed point free group of automorphisms of G. In N we have $1 = e_1 + e_2 + \cdots + e_u$ where the e_i's are mutually orthogonal primitive idempotents. Suppose the positive integer s is a proper divisor of u, say $u = ts$. Let

$$\begin{aligned} f_1 &= e_1 + e_2 + \cdots + e_s \\ f_2 &= e_{s+1} + e_{s+2} + \cdots + e_{2s} \\ &\dots \\ f_t &= e_{(t-1)s+1} + e_{(t-1)s+2} + \cdots + e_{ts}, \end{aligned}$$

then $1 = f_1 + f_2 + \cdots + f_t$ where the f_i's are mutually orthogonal idempotents. If M is an SE-group for N and if m is any element in M then $m = m(f_1 + f_2 + \cdots + f_t) = mf_1 + mf_2 + \cdots + mf_1$. (For if m belongs to M, then for each i,$(m(f_1 + f_2 + \cdots + f_t) - mf_t - \cdots - mf_2 - mf_1)e_i = 0$. Since M is an SE-group, $m(f_1 + f_2 + \cdots + f_t) - mf_t - \cdots - mf_2 - mf_1 = 0$).

Theorem 2. *If N and M are as above then $C(N,M)$ is isomorphic to $C(f_1Nf_1, Mf_1)$.*

Proof. Since N is simple and A is fixed point free then for every i, j such that $i \neq j$ there exist elements e_{ij} in e_iNe_j such that $e_{ij}e_{ji} = e_i$. The proof of Theorem 2 is the same as that of Theorem 1 replacing e_i with f_i, and replacing $e_{1i}e_{i1}$ by

$$f_{1i} = e_{1,(i-1)s+1} + e_{2,(i-1)s+2} + \cdots + e_{s,is},$$
$$f_{i1} = e_{(i-1)s+1,1} + e_{(i-1)s+2,2} + \cdots + e_{is,s},$$

where we have $f_{1i}f_{i1} = f_1$ and $f_{i1}f_{1i} = f_i$. ◇

We mention two special situations for Theorem 2.

(a) *Let $M = N$, then $C(N, N_N)$ is isomorphic to each of the following: $C(e_1Ne_1, Ne_1)$, $C(f_1Nf_1, Nf_1)$ and N.*

(b) *Let $M = N^k$ (k, a positive integer), then $C(N, N^k)$ is isomorphic to each of the following: $C(e_1Ne_1, (Ne_1)^k)$ and $C(f_1Nf_1, (Nf_1)^k)$.*

3. Structure of $C(N,M)$, N not simple

Assume N is a finite semisimple near-ring where $N = N_1 \oplus N_2 \oplus \oplus \cdots \oplus N_s$ (direct sum) with each N_i simple. If f_i is the identity of N_i for each i then $1 = f_1 + f_2 + \cdots + f_s$ in N. Let M be an SE-group for N. Then if m is in M we have $m = m(f_1 + f_2 + \cdots + f_s) = mf_1 + +mf_2 + \cdots + mf_s$.

Theorem 3. *If N and M are as above then $C(N,M) = C(N_1, Mf_1) \oplus \oplus C(N_2, Mf_2) \oplus \cdots \oplus C(N_s, Mf_s)$ (direct sum).*

Note. Since each N_i is simple, it follows that if the conditions of Theorem 1 are satisfied (which will be the case if N_i is not a ring) then $C(N_i, Mf_i)$ is isomorphic to $C(e_i^1N_ie_i^1, Mf_ie_i^1) = C(e_i^1N_ie_i^1, Me_i^1)$ where in N_i, $f_i = e_1^1 + e_i^2 + \cdots + e_i^{t_i}$ (primitive idempotents).

Proof of Theorem 3. Clearly $M = Mf_1 \oplus Mf_2 \oplus \cdots \oplus Mf_s$ (direct sum). If g belongs to $C(N,M)$ then $g(Mf_i)$ is a subset of Mf_i and $g(mf_1 + mf_2 + \cdots + mf_s) = gmf_1 + gmf_2 + \cdots + gmf_s$. The map φ from $C(N,M)$ to $C(N_1, Mf_1) \oplus C(N_2, Mf_2) \oplus \cdots \oplus C(N_s, Mf_s)$ defined by $\varphi(g) = g_1 + g_2 + \cdots + g_s$ is our isomorphism where g_i is g restricted to Mf_i. ◇

The following is valid for an arbitrary finite centralizer near-ring of the form $C(A,G)$ where A is a group of automorphisms of G.

Theorem 4. *Let N be the finite centralizer near-ring $C(A,G)$. If H is an A-invariant subgroup of G let $R = \{f \text{ in } N | f(G) \text{ is a subset of } H\}$. Then $C(N,R)$ is isomorphic to R.*

Proof. R is easily seen to be a right ideal of N. If e_H is the idempotent in N which is the identity on H and 0 off H then $R = e_H N$. We have

$$\begin{aligned} C(N,R) &= \{f | f(rn) = f(r)n \text{ for all } r \text{ in } R \text{ and } n \text{ in } N\} \\ &= \{f | f(e_H n) = f(e_H)n \text{ for all } n \text{ in } N\} \\ &= \{L_r | r \text{ is in } R\} \text{ (where } L_r \text{ is the left multiplication} \\ &\quad \text{map by } r \text{ on } R) \text{ which is ismorphic to } R. \ \Diamond \end{aligned}$$

4. Applications to matrix near-rings

J.D.P. Meldrum and A.P.J. van der Walt have introduced the concept of a matrix near-ring (see [5]) which we now recall. Let N be a near-ring with 1 and let t be a positive integer. For an element r in N and for integers i,j with $1 \leq i,\ j \leq t$ define the function f_{ij}^r on N as follows:

$$f_{ij}^r(n_1, \ldots, n_i, \ldots, n_j, \ldots, n_t) = (0, \ldots, rn_j, \ldots, 0, \ldots, 0)$$

(where rn_j is in the i^{th} position). The $t \times t$ matrix near-ring over $N, M_t(N)$, is the subnearring of Map(N^t) generated by $\{f_{ij}^r | r \text{ is in } N$ and $1 \leq i,\ j \leq t\}$. We note that f_{ij}^r belongs to $C(N,N^t)$. Therefore $M_t(N)$ is a subnearring of $C(N,N^t)$. The following result was proven by van der Walt in [8].

Theorem (van der Walt). *Let N be a finite simple near-ring such that $N = C(A,G)$ where A is a fixed point free group of automorphisms on G. Then $M_t(N)$ is isomorphic to $C(A,G^t)$.*

Our information on SE-groups for a finite simple near-ring N can be used together with van der Walt's theorem to prove a near-ring analogue to a familiar matrix ring result in ring theory.

Theorem 5. *Let N be a finite simple near-ring with $N = C(A,G)$ where A is a fixed point free group of automorphisms on G. Let s and t be positive integers. Then $C(C(N,N^s),C(N,N^s)^t)$ is isomorphic to*

$C(N, N^{st})$.

Proof. Since N is simple we have seen that $C(N, N^s)$ is a simple near-ring and $C(N, N^s)$ is isomorphic to $C(A, G^s)$. Using Corollary 2, $C(C(N, N^s), C(N, N^s)^t)$ is isomorphic ot $C(A, (G^s)^t)$ which is isomorphic to $C(A, G^{st})$ and therefore isomorphic to $C(N, N^{st})$. ♢

Corollary 7. *If N is a finite simple near-ring with $N = C(A, G)$ where A is a fixed point free group of automorphisms on G then $M_t(M_s(N))$ is isomorphic to $M_{ts}(N)$.*

Proof. From van der Walt's theorem $C(A, G^{ts})$ is isomorphic to $M_{ts}(N)$ and $M_t(C(A, G^s)$ is isomorphic to $M_t(M_s(N))$. ♢

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