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# SIMULTANEOUS EXTENSIONS OF PROXIMITIES, SEMI-UNIFORMITIES, CONTIGUITIES AND MEROTOPIES II*

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**Abstract:** Given compatible semi-uniformities (or contiguities, or merotopies) on some subspaces of a closure space, we are looking for a common extension of these structures.

**Notations.** In addition to the notations and conventions introduced in §0 (see in [1]), let $A^2 = A \times A$, $A^r = X \setminus A$ (for $A \subset X$); if $\mathcal{U}$

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is a semi-uniformity then $s\mathcal{U}$ denotes the collection of the symmetric entourages contained by $\mathcal{U}$.

## 2. Extending a family of semi-uniformities in a closure space

### A. WITHOUT SEPARATION AXIOMS

**2.1** If a family of semi-uniformities can be extended in a closure space then the closure is necessarily symmetric; this condition will turn out to be sufficient, too. We are going to construct the finest and the coarsest extension.

**Definitions.** For a family of semi-uniformities in a closure space,

a) Let $\mathcal{U}^0$ be these semi-uniformity on $\dot{X}$ for which the following entourages form a subbase $\mathcal{B}$:

(1) $$U_i^0 = U_i \cup (X^2 \setminus X_i^2) \qquad (i \in I,\ U_i s \in \mathcal{U}_i);$$

(2) $$U_{x,B} = \{x\}^{r2} \cup B^{r2} \qquad (x \in X,\ B \subset X,\ x \notin c(B)).$$

b) Let $\mathcal{U}^1$ consist of the entourges $U$ on $X$ that satisfy the following conditions:

(3) $$Ux \in \mathsf{v}(x) \qquad (x \in X);$$

(4) $$U|X_i \in \mathcal{U}_i \qquad (i \in I).\ \diamondsuit$$

$\mathcal{B}$ is a collection of symmetric entourages, and, assuming $X \neq \emptyset$, $\mathcal{B}$ is non-empty (take $B = \emptyset$ in (2)), so it is indeed a subbase for a semi-uniformity. $\mathcal{U}^0$ would not change if $s\mathcal{U}_i$ were replaced by $\mathcal{U}_i$ in (1). It is straightforward to check that $\mathcal{U}^1$ is a semi-uniformity, too.

Similarly to the convention introduced in §1 for proximities, we shall write, if necessary, $\mathcal{U}^k(c, \mathcal{U}_i)$, or even $\mathcal{U}^k(c, \{\mathcal{U}_i : i \in I\})$. In particular, $\mathcal{U}^k(c) = L$, $\mathcal{U}^k(c, \emptyset)$. $(k = 0, 1)$. Analogous notations will be used for Riesz and Lodato semi-uniformities, as well for merotopies and contiguities.

**Theorem.** *A family of semi-uniformities in a symmetric closure space always has extensions; $\mathcal{U}^0$ is the coarsest and $\mathcal{U}^1$ the finest extension.*

**Proof.** 1° *$\mathcal{U}^0$ is coarser than $\mathcal{U}^1$.* It is enough to show that $\mathcal{B} \subset \mathcal{U}^1$.

$U_i^0 x = X$ if $x \in X_i^r$; otherwise $U_i^0 x = U_i x \cup X_i^r$, which is clearly a $c$-neighbourhood of $x$, since $U_i x \in \mathsf{s}_i(x)$. For $j \in I$,

$$U_i^0|X_j = (U_i|X_{ij}) \cup (X_j^2 \setminus X_{ij}^2) = (U_j|X_{ij}) \cup (X_j^2 \setminus X_{ij}^2)$$

holds with a suitable $U_j \in \mathcal{U}_j$ (by the accordance); hence $U_i^0|X_j \supset U_j$, and so $U_i^0|X_j \in \mathcal{U}_j$. This means that $U_i^0$ satisfies (3) and (4), i.e. $U_i^0 \in \mathcal{U}^1$.

$U_{x,B}y \in \mathsf{v}(y)$ for $y \in X$; this follows from $U_{x,B}x = B^r$ for $y = x$, from $U_{x,B} = X$ for $x \neq y \in B^r$, and from $U_{x,B}y = \{x\}^r$ for $y \in B$, because in the last case $c(\{y\}) \subset c(B)$, $x \notin c(\{y\})$, $y \notin c(\{x\})$. $U_{x,B}$ satisfies (4), too: $U_{x,B}|X_i = X_i^2$ if $x \in X_i^r$, and in case $x \in X_i$ we can choose a $U_i \in s\mathcal{U}_i$ with $U_i x \cap (B \cap X_i) = \emptyset$ (as $x \notin c(B) \supset c_i(B \cap X_i)$), and then $U_i \subset U_{x,B}|X_i$.

2° *$\mathcal{U}^1|X_i$ is coarser than $\mathcal{U}_i$.* This is evident from (4).

3° *$\mathcal{U}_i$ is coarser than $\mathcal{U}^0|X_i$.* If $U_i \in \mathcal{U}_i$ then $U_i = U_i^0|X_i \in \mathcal{B}|X_i \subset \subset \mathcal{U}^0|X_i$.

4° *$c(\mathcal{U}^1)$ is coarser than $c$.* This is clear from (3).

5° *$c$ is coarser than $c(\mathcal{U}^0)$.* Observe that

$$\mathsf{v}(x) = \{U_{x,B}x : x \notin c(B)\}. \tag{5}$$

6° *$\mathcal{U}^0$ and $\mathcal{U}^0$ are extensions.* It follows from 1°, 4° and 5° that $\mathcal{U}^0$ and $\mathcal{U}^1$ are compatible, respectively from 1°, 2° and 3° that $\mathcal{U}^0|X_i = = \mathcal{U}_i = \mathcal{U}^1|X_i$.

7° *$\mathcal{U}^0$ is the coarsest extension.* Let $\mathcal{U}$ be another extension; we have to show that $\mathcal{B} \subset \mathcal{U}$.

For $U_i \in s\mathcal{U}_i$, choose $U \in \mathcal{U}$ such that $U|X_i = U_i$; now $U \subset U_i^0$, thus $U_i^0 \in \mathcal{U}$. If $x \notin c(B)$ then $Ux \cap B = \emptyset$ for some $U \in s\mathcal{U}$, and therefore $U \subset U_{x,B}$.

8° *$\mathcal{U}^1$ is the finest extension.* If $\mathcal{U}$ is another extension then each $U \in \mathcal{U}$ satisfies (3), because $\mathcal{U}$ is compatible, and (4), because $\mathcal{U}|X_i = = \mathcal{U}_i$. Hence $\mathcal{U} \subset \mathcal{U}^1$. ◇

**2.2** a) Formulas analogous to 1.3 (1) and (2) are valid for semi-uniformities (and also for merotopies and contiguities). The proofs are essentially the same as the ones given in 1.3 for proximities. We are going to set out the categorical background of these formulas.

Let **C** and **D** be topological categories, and $F : \mathbf{D} \to \mathbf{C}$ a concrete functor. (In contrast to the situation outlined in the introduction of Part I, it is not necessary to assume here that $F$ commutes with the restriction to subsets.) We denote the **C**-structures by $c$, and the **D**-structures by $d$ (with indices when necessary), and use the conventions introduced in §0, except that a family of **D**-structures in a **C**-space is not required to be either compatible or accordant; in particular, $F(d)$ will also be written as $c(d)$. $c < c'$ denotes that $c$ is coarser than $c'$.

Consider a family of **D**-structures in a **C**-space. We say that the **D**-structure $d$ on $X$ is a

*00-overextension* if it is just an extension;
*01-overextension* if $c = c(d)$ and $d_i < d|X_i \quad (i \in I)$;
*10-overextension* if $c < c(d)$ and $d_i = d|X_i \quad (i \in I)$;
*11-overextension* if $c < c(d)$ and $d_i < d|X_i \quad (i \in I)$.

A *pq-underextension* $(p, q = 0, 1)$ is defined in the same way, replacing $<$ by $>$.

Let now a non-empty family of **D**-structures in a **C**-space be fixed.

1° If $d^0[i]$ is the coarsest $pq$-overextension of $\{d_i\}$, and there exists a $pq$-overextension of the whole family then $\sup_i d^0[i]$ is the coarsest $pq$-overextension. (The proof is straightforward.)

2° Assume that (i) the empty family in $(X, c)$ has a coarsest $pq$-overextension $d^0(c)$ (if $p = 0$ then this means that $d^0(c)$ is the coarsest compatible structure); (ii) each $\{d_i\}$ has a coarsest $1q$-overextension $d^{00}[i]$ *with respect to the indiscrete* **C**-*structure on* $X$; (iii) the whole family has a $pq$-overextension. Then

$$\sup\{d^0(c), \sup_i d^{00}[i]\} \tag{1}$$

is the coarsest $pq$-overextension. (The statement is more symmetrical than it looks to be: $d^0(c)$ is the coarsest $p1$-overextension if each $d_i$ is replaced by the indiscrete **D**-structure on $X_i$.)

3° The analogue of 1° is valid for $pq$-underextensions.

4° In the analogue of 2° for $pq$-underextensions, the condition corresponding to (ii) is superfluous, since $d^{11}[i]$ always exists in a topological category: take the coproduct of $d_i$ and the discrete structure on $X_i^r$. (The reason for the difference is that 2° and 4° are not dual: subspaces have not been replaced by quotient spaces.)

Observe that Definition 2.1 gives $\mathcal{U}^0$ in the form (1), and $\mathcal{U}^1$ similarly as an infimum.

b) It is possible to deduce Theorems 1.1 and 1.2 from Theorem 2.1; this will be discussed in Part III, where a result on extending semi-uniformities in a proximity space will enable us to do the converse, too, i.e. to partly prove Theorem 2.1 in two steps, first extending the proximities $\delta(\mathcal{U}_i)$ in $(X, c)$, and then the semi-uniformities in $(X, \delta^0)$ or $(X, \delta^1)$.

## B. RIESZ SEMI-UNIFORMITIES IN A CLOSURE SPACE

**2.3** If a family of semi-uniformities in a closure space has a Riesz extension then each semi-uniformity is Riesz, the closure is weakly separated, and the trace filters are Cauchy (because the neighbourhood filters in a Riesz semi-uniform space are Cauchy). We are going to prove that these conditions are sufficient, too.

**Definition.** For a family of semi-uniformities in a closure space, let

$$\mathcal{U}^1_R = \{U \in \mathcal{U}^1 : \Delta \subset \operatorname{Int} U\}. \diamondsuit \tag{1}$$

$\mathcal{U}^1_R$ is clearly a semi-uniformity.

**Theorem.** *A family of semi-uniformities in a weakly separated closure space has a Riesz extension iff the trace filters are Cauchy; if so then $\mathcal{U}^0$ is the coarsest and $\mathcal{U}^1_R$ the finest Riesz extension.*

**Proof.** Assume that the trace filters are Cauchy.

1° *$\mathcal{U}^0$ is coarser than $\mathcal{U}^1_R$.* In view of 1° from the proof of Theorem 2.1, it is enough to show that $\Delta \subset \operatorname{Int} U$ holds for each $U \in \mathcal{B}$.

$\Delta \subset \operatorname{Int} U^0_i$, because for $x \in X$, there is an $A \in \mathsf{s}_i(x)$ such that $A^2 \subset U_i$, and then $B = A \cup X^r_i \in \mathsf{v}(x)$, thus $(x, x) \in \operatorname{Int} B^2$ and $B^2 \subset U^0_i$.

Similarly, $\Delta \subset \operatorname{Int} U_{x,B}$, because for $y \in X$, there is an $A \in \mathsf{v}(y)$ such that $A^2 \subset U_{x,B}$, namely

$$A = \begin{cases} B^r & \text{if } y \in c(\{x\}), \\ \{x\}^r & \text{if } y \notin c(\{x\}). \end{cases}$$

(If $y \in c(\{x\})$ then, $c$ being weakly separated, from $x \notin c(B)$ we have $y \notin c(B)$, thus $B^r \in \mathsf{v}(y)$ indeed.)

2° *$\mathcal{U}^1_R$ is a Riesz extension.* By 1°, the evident statement $\mathcal{U}^1_R \subset \mathcal{U}^1$, and Theorem 2.1, $\mathcal{U}^1_R$ is an extension. The compatibility of $\mathcal{U}^1_R$ implies that it is Riesz, as Int in (1) is now the $c(\mathcal{U}^1_R) \times c(\mathcal{U}^1_R)$-interior.

3° $\mathcal{U}^0$ is Riesz, too, because it is coarser than a Riesz semi-uniformity inducing the same closure. Given a Riesz extension $\mathcal{U}$, we have $\mathcal{U} \subset \mathcal{U}^1$ by Theorem 2.1, and the elements of $\mathcal{U}$ satisfy (1), thus $\mathcal{U} \subset \mathcal{U}^1_R$. On the other hand $\mathcal{U}^0 \subset \mathcal{U}$, again by Theorem 2.1. $\diamondsuit$

If $\{\text{int}X_i : i \in I\}$ covers $X$ and each $\mathcal{U}_i$ is Riesz then it is not necessary to assume that the trace filters are Cauchy: For $x \in X$ and $U_i \in \mathcal{U}_i$, take $j \in I$ with $x \in \text{int}X_j$, and $U_j \in \mathcal{U}_j$ such that $U_j|X_{ij} = U_i|X_{ij}$. As $\mathcal{U}_j$ is Riesz, there is an $A \in \mathsf{s}_j(x)$ with $A^2 \subset U_j$. $x \in \text{int}X_j$ implies that $A \in \mathsf{v}(x)$, thus $A \cap X_i \in \mathsf{s}_i(x)$, and $(A \cap X_i)^2 \subset U_i$.

**Corollary.** *A family of Riesz-semi-uniformities in a weakly separated closure space has a Riesz extension iff* $\mathcal{U}_i \subset \mathcal{U}^1_R(c)|X_i$ $(i \in I)$.

**Proof.** Just like the proof of Corollary 1.4. ◇

## C. LODATO SEMI-UNIFORMITIES IN A CLOSURE SPACE

**2.4** If a family of semi-uniformities in a closure space has a Lodato extension then each semi-uniformity is Lodato, the closure is an $S_1$-topology, and the trace filters are Cauchy. A modification of Example 1.8 shows that these conditions are not sufficient: replace $\delta_0$ by the Euclidean uniformity $\mathcal{U}_0$ on $X_0$, and $\delta^1_R(c)$ by a Lodato semi-uniformity $\mathcal{V}$ compatible with it (e.g. by $\mathcal{U}^1_R(c)$); now $\mathcal{U}_0$ and $\mathcal{U}_1 = \mathcal{V}|X_1$ satisfy the necessary conditions, but if they had a Lodato extension $\mathcal{U}$ then the Lodato proximity $\delta(\mathcal{U})$ would extend $\delta_0$ and $\delta_1$. In Example 2.10, we shall define Lodato semi-uniformities in a closure space that do not have a Lodato extension, although the Lodato proximities induced by them do have one.

**Notation.** In a closure space $(X, c)$, put

$$V_{x,B} = V_{x,B;X} = c(\{x\})^{r2} \cup c(B)^{r2} \tag{1}$$

for $x \in X$, $B \subset X$, $x \notin c(B)$. ◇

**Lemma.** *If $c$ is weakly separated then* $V_{x,B} = \text{Int}\, U_{x,B}$*; so if $\mathcal{U}$ is a compatible Lodato semi-uniformity then* $V_{x,B} \in \mathcal{U}$.

**Proof.** $V_{x,B} \subset \text{Int}\, U_{x,B}$ is evident. Conversely, let $(y, z) \in \text{Int}\, U_{x,B}$. If $y, z \notin c(\{x\})$ then clearly $(y, z) \in V_{x,B}$. If, say, $y \in c(\{x\})$ then take $M, N$ such that $y \in \text{int}\, M$, $z \in \text{int}\, N$, and $M \times N \subset U_{x,B}$. Now $x \in M$ implies $N \subset B^r$, thus $z \in c(B)^r$. On the other hand, $y \in c(B)^r$ follows from the weak separatedness. Hence $(y, z) \in V_{x,B}$ again.

The second statement follows from the first one, using Theorem 2.1 applied to $I = \emptyset$. ◇

**2.5 Definition.** For a family of Lodato semi-uniformities in an $S_1$-space, let

(1) $$\mathcal{U}_L^1 = \{U \in \mathcal{U}^1 : \operatorname{Int} U \in \mathcal{U}^1\}.$$

In other words, the $c \times c$-open elements of $\mathcal{U}^1$ form a base for $\mathcal{U}_L^1$. ◇

**Lemma.** *For a family of Lodato semi-uniformities in an $S_1$-space, $\mathcal{U}_L^1$ is a compatible Lodato semi-uniformity; it is the finest one among those Lodato semi-uniformities $\mathcal{U}$ on $X$ that induce a closure coarser than $c$, and for which $\mathcal{U}|X_i$ is coarser than $\mathcal{U}_i$ ($i \in I$).*

**Proof.** $\mathcal{U}_L^1$ is clearly a semi-uniformity. $\mathcal{U}_L^1 \subset \mathcal{U}^1$, so it follows from Theorem 2.1 that $\mathcal{U}_L^1|X_i$ is coarser than $\mathcal{U}_i$ and $c(\mathcal{U}_L^1)$ is coarser than $c$.

1° *$c(\mathcal{U}_L^1)$ is finer than $c$.* It suffices to see that $U_{x,B} \in \mathcal{U}_L^1$ ($x \in X$, $B \subset X$, $x \notin c(B)$). $V_{x,B}$ is clearly a $c \times c$-open entourage contained by $U_{x,B}$, so we have only to check that $V_{x,B} \in \mathcal{U}^1$.

2.1 (3) is satisfied, since $V_{x,B}y$ is open for $y \in X$.

To prove 2.1 (4), fix an $i \in I$. If $c(\{x\}) \cap X_i = \emptyset$ then $X_i^2 \subset V_{x,B}$, thus $V_{x,B}|X_i \in \mathcal{U}_i$ is now evident. Otherwise, pick a point $y \in c(\{x\}) \cap \cap X_i$; then ($c$ being an $S_1$-topology) $c(\{y\}) = c(\{x\})$ and $y \notin c(B) \supset \supset A = c(B) \cap X_i$. As $A$ is $c_i$-closed, Lemma 2.4 gives

$$V_{y,A;X_i} = (X_i \setminus c_i(\{y\}))^2 \cup (X_i \setminus A)^2 \in \mathcal{U}_i.$$

Now $V_{x,B}|X_i = V_{y,A;X_i}$ follows from $c_i(\{y\}) = c(\{y\}) \cap X_i = c(\{x\}) \cap \cap X_i$. Thus $V_{x,B}|X_i \in \mathcal{U}_i$ again.

2° *$\mathcal{U}_L^1$ is Lodato.* We have established that $\mathcal{U}_L^1$ is compatible, so it is Lodato by (1) (since $c$ is a topology).

3° *$\mathcal{U}_L^1$ is finest.* Let $\mathcal{U}$ be another Lodato semi-uniformity with $\mathcal{U}|X_i \subset \mathcal{U}_i$ ($i \in I$) and $c(\mathcal{U})$ coarser than $c$; we have to show that $\mathcal{U} \subset \mathcal{U}_L^1$. $\mathcal{U} \subset \mathcal{U}^1$ is evident; moreover, $\mathcal{U}$ has a base consisting of $c(\mathcal{U}) \times c(\mathcal{U})$-open entourages, which are then $c \times c$-open, too. ◇

**2.6 Definition.** For a family of Lodato semi-uniformities in an $S_1$-space, let $\mathcal{U}_L^0$ be the filter on $X^2$ generated by the subbase $\mathcal{B}_L$ consisting of the following sets:

$$\begin{array}{ll} \operatorname{Int} U_i^0 & (i \in I,\ U_i \in s\mathcal{U}_i); \\ V_{x,B} & (B \subset X,\ x \in c(B)^r). \ \diamond \end{array}$$

The elements of $\mathcal{B}_L$ are symmetric, thus $\mathcal{U}_L^0$ is a semi-uniformity iff each $\operatorname{Int} U_i^0$ is an entourage, i.e. iff the trace filters are Cauchy. It does not change $\mathcal{U}_L^0$ iff $s\mathcal{U}_i$ is replaced by $\mathcal{U}_i$ and/or $V_{x,B}$ by $c(\{x\})^{r2} \cup B^{r2}$. Observe that

(1) $$\mathcal{B}_L = \{\operatorname{Int} U : U \in \mathcal{B}\},$$

$\{\operatorname{Int} U \in \mathcal{U}^0\}$ is a base for $\mathcal{U}_L^0$, and $\mathcal{U}^0 \subset \mathcal{U}_L^0$.

**Lemma.** *If a family of Lodato semi-uniformities is given in an $S_1$-space, and the trace filters are Cauchy then $\mathcal{U}_L^0$ is the coarsest one among those compatible Lodato semi-uniformities $\mathcal{U}$ on $X$ for which $\mathcal{U}|X_i$ is finer than $\mathcal{U}_i$ $(i \in I)$.*

**Proof.** Theorem 2.1 and $\mathcal{U}^0 \subset \mathcal{U}_L^0$ imply that $\mathcal{U}_L^0|X_i$ is finer than $\mathcal{U}_i$ and $c(\mathcal{U}_L^0)$ is finer than $c$. $c(\mathcal{U}_L^0)$ is also coarser than $c$, since the elements of the subbase $\mathcal{B}_L$ are open; hence $\mathcal{U}_L^0$ is compatible and Lodato.

Let $\mathcal{U}$ be another compatible Lodato semi-uniformity with $\mathcal{U}_i \subset \subset \mathcal{U}|X_i$. Now $U_i^0 \in \mathcal{U}$, and so $\operatorname{Int} U_i^0 \in \mathcal{U}$ (as $\mathcal{U}$ is Lodato). $V_{x,B} \in \mathcal{U}$ by Lemma 2.4. ◇

**2.7 Lemma.** *A family of Lodato semi-uniformities in an $S_1$-space has a Lodato extension iff $\mathcal{U}_L^0 \subset \mathcal{U}_L^1$; if so then both $\mathcal{U}_L^0$ and $\mathcal{U}_L^1$ are Lodato extensions.*

**Proof.** Lemmas 2.5 and 2.6, using that if $\mathcal{U}_L^0 \subset \mathcal{U}_L^1$ then the elements of $\mathcal{U}_L^0$ are entourages, thus the trace filters are Cauchy. ◇

**Theorem.** *A family of Lodato semi-uniformities in an $S_1$-space has a Lodato extension iff the trace filters are Cauchy, and for any $i, j \in I$,*

$$(1) \qquad (\operatorname{Int} U_i^0)|X_j \in \mathcal{U}_j \qquad (U_i \in \mathcal{U}_i);$$

*if so then $\mathcal{U}_L^0$ is the coarsest and $\mathcal{U}_L^1$ is the finest Lodato extension.*

*Remark.* The accordance can be written in the following equivalent form: $U_i^0|X_j \in \mathcal{U}_j$ for $i, j \in I$, $U_i \in \mathcal{U}_i$, of which (1) is clearly a strengthening. For $i = j$, (1) is equivalent to the statement that $\mathcal{U}_i$ is Lodato, so it was in fact superfluous to assume that the semi-uniformities are Lodato.

**Proof.** The necessity is obvious. By Lemma 2.6, the sufficiency will also follow if we show that $\mathcal{U}_L^0|X_j \subset \mathcal{U}_j$, i.e. that $\mathcal{B}_L|X_j \subset \mathcal{U}_j$ $(j \in I)$. For $\operatorname{Int} U_i^0$, this is just (1). $V_{x,B} \in \mathcal{U}^1$ was checked in 1° of the proof of Lemma 2.5, so $V_{x,B}|X_j \in \mathcal{U}_j$ by Theorem 2.1. The remaining statements follow from Lemmas 2.7, 2.6 and 2.5. ◇

**Corollary.** *A family of semi-uniformities in an $S_1$-space has a Lodato extension iff $\{\mathcal{U}_i, \mathcal{U}_j\}$ has a Lodato extension for any $i, j \in I$.* ◇

**2.8 Corollary.** *A single Lodato semi-uniformity given in an $S_1$-space has a Lodato extension iff the trace filters are Cauchy.* ◇

**2.9 Theorem.** *Let a family of Lodato semi-uniformities be given in*

*an $S_1$-space. Assume that either each $X_i$ is open and the trace filters are Cauchy or each $X_i$ is closed. Then there exists a Lodato extension.*
**Proof.** We are going to check that if $U_i \in \mathcal{U}_i$ is open then

$$\text{(2)} \qquad \operatorname{Int} U_i^0 \supset \operatorname{Int}_j (U_i^0|X_j);$$

this is sufficient for 2.7 (1), because the open entourages form a base for $\mathcal{U}_i$ (as $\mathcal{U}_i$ is Lodato), and the right hand side of (1) belongs to $\mathcal{U}_j$ (as the semi-uniformities are accordant, and $\mathcal{U}_j$ is Lodato). Take $(x,y)$ from the right hand side of (1).

1° Let $X_i$ be closed. If $x, y \in X_{ij}$ then, $U_i$ being open, we can pick $c$-open sets $A \ni x$ and $B \ni y$ such that $(A \times B)|X_i \subset U_i$, which implies that $A \times B \subset U_i^0$; thus $(x,y) \in \operatorname{Int} U_i^0$. If, say, $x \in X_j \setminus X_i$ then $(x,y) \in X_i^r \times X$, which is a $c \times c$-open set contained by $U_i^0$.

2° If $X_j$ is open then there are $c$-open sets $A \ni x$ and $B \ni y$ such that $A, B \subset X_j$ and $A \times B \subset U_i^0|X_j \subset U_i^0$. ◇

The analogue of Theorem 1.13 is not valid for semi-uniformities (although it holds for merotopies and contiguities, see Theorems 3.8 and 4.5), not even under the stronger assumption

$$\text{(2)} \qquad c(X_i \setminus X_j) \cap c(X_j \setminus X_i) = \emptyset \qquad (i,j \in I):$$

Example. Let $H = ]0,1[$ , $T = \{0\} \cup \{1/n : n \in \mathbb{N}\}$,

$$X = (T \cup [2,3[) \times H, X_0 = X \setminus (\{0\} \times H), X_1 = X \setminus (]2,3[\times H).$$

Let $c$ be the Euclidean topology on $X$, and $\{U_i(\varepsilon) : \varepsilon > 0\}$ a base for $\mathcal{U}_i$ on $X_i$ ($i = 0,1$), where, with $P \otimes Q$ denoting $(P \times Q) \cup (Q \times P)$,

$$U_0(\varepsilon) = U(\varepsilon)|X_0 \cup \bigcup_{n \in \mathbb{N}} ((\{1/n\} \times ]0,\varepsilon[) \otimes ([2, 2+1/n[ \times ]0,\varepsilon[)),$$
$$U_1(\varepsilon) = U(\varepsilon)|X_1 \cup ((T \times ]0,\varepsilon[) \otimes (\{2\} \times ]0,\varepsilon[)),$$

and, for $x, y \in X$, $x\, U(\varepsilon)\, y$ iff the Euclidean distance of $x$ and $y$ is $< \varepsilon$. $\{\mathcal{U}_0, \mathcal{U}_1\}$ is a family of Lodato semi-uniformities in $(X,c)$, the trace filters are Cauchy, and (2) holds. But there is no Lodato extension: 2.7 (1) fails for $i = 0$, $j = 1$, $U_i = U_0(1)$, since

$$((0,\varepsilon/2),(2,\varepsilon/2)) \in U_1(\varepsilon) \setminus (\operatorname{Int} U_0(1)^0)|X_1. \ \diamond$$

**2.10** By Theorem 1.13, the induced proximities have a Lodato extension in the above example. We can give, however, a much simpler example with this property:
**Example.** With $X, X_0, X_1$ and $c$ as in Example 1.8, let $\mathcal{U}_1$ be the

Euclidean uniformity on $X_1$, $\mathcal{U}_0$ the precompact uniformity compatible with the Euclidean proximity on $X_0$. Now the trace filters are Cauchy, and the induced proximities have a Lodato extension (namely the Euclidean proximity on $X$), but $\mathcal{U}_0$ and $\mathcal{U}_1$ do not have one, since 2.7 (1) fails for $i = 1$ and $j = 0$. ◊

**2.11** Concerning extensions of a single *uniformity*, see [6], [5], [7], [2] §5, [3], [4] §2. The same can be said about simultaneous extensions of uniformities as in the case of Efremovich proximities, cf. 1.16; see [4] Remark 1.13 c) and Example 1.13 b) for details.

## 3. Extending a family of merotopies in a closure space

### A. WITHOUT SEPARATION AXIOMS

**3.1** If a family of merotopies can be extended in a closure space then the closure is symmetric; this condition will be proved to be sufficient, too. Definitions, results and proofs are very similar to those in §2.

**Definitions.** For a family of merotopies in a closure space,

a) Let $\mathsf{M}^0$ be the merotopy on $X$ for which the following covers form a subbase $\mathsf{B}$:

$$(1)\qquad \mathsf{c}_i^0 = \{C_i^0 = C_i \cup X_i^r : C_i \in \mathsf{c}_i\} \qquad (i \in I, \mathsf{c}_i \in \mathsf{M}_i);$$

$$(2)\qquad \mathsf{c}_{x,B} = \{\{x\}^r, B^r\} \qquad (B \subset X, x \in c(B)^r).$$

b) Let $\mathsf{M}^1$ consist of the covers $\mathsf{c}$ of $X$ that satisfy the following conditions:

$$(3)\qquad \mathrm{St}\,(x, \mathsf{c}) \in \mathsf{v}(x) \qquad (x \in X);$$

$$(4)\qquad \mathsf{c}|X_i \in \mathsf{M}_i \qquad (i \in I)\ \Diamond.$$

**Theorem.** *A family of merotopies in a symmetric closure space always has extensions;* $\mathsf{M}^0$ *is the coarsest and* $\mathsf{M}^1$ *the finest extension.*

**Proof.** 1° $\mathsf{M}^0$ *is coarser than* $\mathsf{M}^1$. It is enough to show that $\mathsf{B} \subset \mathsf{M}^1$.

If $x \in X_i^r$ then $\mathrm{St}(x, \mathsf{c}_i^0) = X \in \mathsf{v}(x)$; otherwise $\mathrm{St}(x, \mathsf{c}_i^0) = \mathrm{St}(x, \mathsf{c}_i) \cup X_i^r \in \mathsf{v}(x)$, since $\mathrm{St}(x, \mathsf{c}_i) \in \mathsf{s}_i(x)$. It follows easily from the accordance that $\mathsf{c}_i^0$ satisfies (4), too. Thus $\mathsf{c}_i^0 \in \mathsf{M}^1$.

$\mathrm{St}(y, \mathsf{c}_{x,B})$ is equal to $B^r$ if $y = x$, to $X$ if $x \neq y \in B^r$, and to

$\{x\}^r$ if $y \in B$; thus it belongs to $\mathsf{v}(y)$, in the last case by the symmetry of $c$. $\mathsf{c}_{x,B}$ satisfies (4), too: if $x \in X_i^r$ then $\{X_i\} \in \mathsf{c}_{x,B}|X_i$; otherwise pick $\mathsf{c}_i \in \mathsf{M}_i$ with $\mathrm{St}(x, \mathsf{c}_i) \cap B = \emptyset$, and then $\mathsf{c}_i$ refines $\mathsf{c}_{x,B}|X_i$.

2° $\mathsf{M}^0$ *and* $\mathsf{M}^1$ *are extensions.* Just like in the proof of Theorem 2.1, replacing 2.1 (5) by

$$\mathsf{v}(x) = \{\mathrm{St}(x, \mathsf{c}_{x,B}) : x \notin c(B)\}. \tag{5}$$

3° $\mathsf{M}^0$ *is coarsest,* $\mathsf{M}^1$ *is finest.* Check that if $\mathsf{M}$ is an extension then $\mathsf{B} \subset \mathsf{M}$, and, on the other hand, each $\mathsf{c} \in \mathsf{M}$ satisfies (3) and (4). ◇

## B. RIESZ MEROTOPIES IN A CLOSURE SPACE

**3.2** If a family of merotopies in a closure space has a Riesz extension then the merotopies are Riesz, the closure is weakly separated, and the trace filters are Cauchy. These conditions are also sufficient.

**Definition.** For a family of merotopies in a closure space, let

$$\mathsf{M}^1_R = \{\mathsf{c} \in \mathsf{M}^1 : \mathrm{int}\,\mathsf{c} \text{ is a cover of } X\}. \tag{1}$$

Observe that

$$\mathrm{int}\,\mathsf{c}_{x,B} = \{c(\{x\})^r, c(B)^r\}. \tag{2}$$

**Theorem.** *A family of merotopies in a weakly separated closure space has a Riesz extension iff the trace filters are Cauchy; if so then* $\mathsf{M}^0$ *is the coarsest and* $\mathsf{M}^1_R$ *the finest Riesz extension.*

**Proof.** Assume that the trace filters are Cauchy.

1° $\mathsf{M}^0$ *is coarser than* $\mathsf{M}^1_R$. By $\mathsf{M}^0 \subset \mathsf{M}^1$, it is enough to show that $\mathrm{int}\,\mathsf{c}$ is a cover for $\mathsf{c} \in \mathcal{B}$.

$\mathrm{int}\,\mathsf{c}_i^0$ is a cover, because the trace filters are Cauchy. $\mathrm{int}\,\mathsf{c}_{x,B}$ is also a cover, since $c$ is weakly separated, and so $c(\{x\}) \cap c(B) = \emptyset$.

2° The remaining statements can be proved in the same way as in 2° and 3° in the proof of Theorem 2.3, replacing entourages by covers and Int by int. ◇

**Corollary.** *A family of Riesz merotopies in a weakly separated closure space has a Riesz extension iff* $\mathsf{M}_i \subset \mathsf{M}^1_R(c)|X_i$ $(i \in I)$. ◇

## C. LODATO MEROTOPIES IN A CLOSURE SPACE

**3.3** If a family of merotopies in a closure space has a Lodato extension then the merotopies are Lodato, the closure is an $\mathrm{S}_1$-topology, and the trace filters are Cauchy. Example 1.8 can be modified for merotopies in the same way as for semi-uniformities (cf. 2.4) showing that the above conditions are not sufficient; a better example will be given in 3.8.

**Notation.** $\mathsf{d}_{x,B} = \mathsf{d}_{x,B;X} = \operatorname{int} \mathsf{c}_{x,B}$ for $B \subset X$ and $x \in c(B)^r$ (cf. 3.2 (2)). ◇

**Lemma.** *If* $\mathsf{M}$ *is a compatible Lodato merotopy then* $\mathsf{d}_{x,B} \in \mathsf{M}$. ◇

**3.4 Definition.** For a family of Lodato merotopies in an $S_1$-space, let

$$\mathsf{M}^1_L = \{\mathsf{c} \in \mathsf{M}^1 : \operatorname{int} \mathsf{c} \in \mathsf{M}^1\}.$$

In other words, the open covers contained by $\mathsf{M}^1$ form a base for $\mathsf{M}^1_L$. ◇

**Lemma.** *For a family of Lodato merotopies in an* $S_1$*-space,* $\mathsf{M}^1_L$ *is a compatible Lodato merotopy; it is the finest one among those Lodato merotopies* $\mathsf{M}$ *on* $X$ *that induce a closure coarser than* $c$*, and for which* $\mathsf{M}|X_i$ *is coarser than* $\mathsf{M}_i$ $(i \in I)$.

**Proof.** The argument runs along the same lines as the proof of Lemma 2.5, therefore we confine ourselves to showing that $c(\mathsf{M}^1_L)$ is finer than $c$. It is enough to see that $\mathsf{d}_{x,B} \in \mathsf{M}^1$, because then $\mathsf{c}_{x,B} \in \mathsf{M}^1_L$, and 3.1 (5) can be applied. 3.1 (3) is satisfied, since $\mathsf{d}_{x,B}$ is an open cover.

If $c(\{x\}) \cap X_i = \emptyset$ then $\{X_i\} \in \mathsf{d}_{x,B}|X_i$ thus $\mathsf{d}_{x,B}|X_i \in \mathsf{M}_i$. Otherwise, pick a point $y \in c(\{x\}) \cap X_i$. Now $c(\{x\}) = c(\{y\})$, thus $\mathsf{d}_{x,B} = \mathsf{d}_{y,B}$. But

$$\mathsf{d}_{y,B}|X_i = \mathsf{d}_{y,c(B)}|X_i = \mathsf{d}_{y,c(B)\cap X_i;X_i} \in \mathsf{M}_i$$

by Lemma 3.3. So $\mathsf{d}_{x,B}|X_i \in \mathsf{M}_i$ again, i.e. 3.1 (4) is fullfilled, too. ◇

**3.5 Definition.** Given a family of Lodato merotopies in an $S_1$-space such that the trace filters are Cauchy, let $\mathsf{M}^0_L$ be the merotopy on $X$ for which the following covers form a subbase $\mathsf{B}_L$:

$$\operatorname{int} \mathsf{c}^0_i \qquad (i \in I, \mathsf{c}_i \in \mathsf{M}_i);$$
$$\mathsf{d}_{x,B} \qquad (B \subset X, x \in c(B)^r). \text{ ◇}$$

$\mathsf{B}_L$ is indeed a subbase for a merotopy ($\operatorname{int} \mathsf{c}^0_i$ is a cover, because the trace filters are Cauchy; this condition could be dropped as in Definition 2.6, but then the notion of a subbase had to be generalized from covers to arbitrary collections). We have $\mathsf{B}_L = \{\operatorname{int} \mathsf{c} : \mathsf{c} \in \mathsf{B}\}$. $\{\operatorname{int} \mathsf{c} : \mathsf{c} \in \mathsf{M}^0\}$ is a base for $\mathsf{M}^0_L$.

**Lemma.** *If a family of Lodato merotopies is given in an* $S_1$*-space, and the trace filters are Cauchy then* $\mathsf{M}^0_L$ *is the coarsest one among those* *compatible Lodato merotopies* $\mathsf{M}$ *on* $X$ *for which* $\mathsf{M}|X_i$ *is finer than* $\mathsf{M}_i$ $(i \in I)$.

**Proof.** Similar to the proof of Lemma 2.6. ◇

**3.6 Lemma.** *A family of Lodato merotopies in an* $S_1$*-space has a Lodato extension iff the trace filters are Cauchy and* $\mathsf{M}_L^0 \subset \mathsf{M}_L^1$*; if so then both* $\mathsf{M}_L^0$ *and* $\mathsf{M}_L^1$ *are Lodato extensions.* ◇

**Theorem.** *A family of Lodato merotopies in an* $S_1$*-space has a Lodato extension iff the trace filters are Cauchy, and, for any* $i, j \in I$,

$$(1) \qquad (\operatorname{int} \mathsf{c}_i^0)|X_j \in \mathsf{M}_j \qquad (\mathsf{c}_i \in \mathsf{M}_i);$$

*if so then* $\mathsf{M}_L^0$ *is the coarsest and* $\mathsf{M}_L^1$ *is the finest Lodato extension.*

*Remark.* The accordance of merotopies can be written in the following form: $\mathsf{c}_i^0|X_j \in \mathsf{M}_j$.

**Proof.** Similar to the proof of Theorem 2.7, using that $\mathsf{d}_{x,B} \in \mathsf{M}^1$ was established in the proof of Lemma 3.4. ◇

**Corollary.** *A family of merotopies in an* $S_1$*-space has a Lodato extension iff* $\{\mathsf{M}_i, \mathsf{M}_j\}$ *has a Lodato extension for any* $i, j \in I$. ◇

**3.7 Corollary.** *A single Lodato merotopy in an* $S_1$*-space has a Lodato extension iff the trace filters are Cauchy.* ◇

**3.8 Theorem.** *Let a family of Lodato merotopies be given in an* $S_1$*-space, assume that the trace filters are Cauchy, and*

$$(1) \qquad c(X_i \setminus X_j) \cap (X_j \setminus X_i) = \emptyset \qquad (i, j \in I).$$

*Then there exists a Lodato extension.*

**Proof.** To prove 3.6 (1), it is enough to show that if $\mathsf{c}_i \in \mathsf{M}_i$ is open (which may be assumed, as $\mathsf{M}_i$ is Lodato) then $\operatorname{int}_j(\mathsf{c}_i^0|X_j)$ is a refinement of $(\operatorname{int} \mathsf{c}_i^0)|X_j$, because the former belongs to $\mathsf{M}_j$ by the accordance and the Lodato property of $\mathsf{M}_j$. The above statement is a consequence of

$$(2) \qquad \operatorname{int}_j(G_i^0 \cap X_j) \subset \operatorname{int} G_i^0,$$

where $G_i$ is $c_i$-open.

For the proof of (2), take a point $x$ from the left hand side of it. If $x \in X_{ij}$ then $x \in G_i$, implying $x \in \operatorname{int} G_i^0$. If $x \in X_j \setminus X_i$ then pick a $c$-open set $H$ such that $x \in H$ and $H \cap X_j \subset G_i^0$; we may assume by (1) that $H \cap (X_i \setminus X_j) = \emptyset$, thus $H \cap X_i \subset G_i^0$, implying $H \subset G_i^0$. ◇

**Corollary.** *Let a family of Lodato merotopies be given in an* $S_1$*-space.*

*Assume that either each $X_i$ is open and the trace fileters are Cauchy or each $X_i$ is closed. Then there exists a Lodato extension.* $\diamondsuit$

**Example.** Take $S = \{1/n : n \in \mathbb{N}\}$, $X = S \times (\{0\} \cup S)$, $X_0 = S \times \{0\}$, $X_1 = X_0^r$. Let $c$ be the Euclidean topology (inherited from $\mathbb{R}^2$) on $X$, and $\mathsf{M}_0$ the merotopy on $X_0$ that consists of all the covers containing at least one cofinite set. For $\varepsilon > 0$, consider the cover

$$\begin{aligned}(3)\qquad \mathfrak{c}_1(\varepsilon) = \{(]p, p+\varepsilon[\times]q, q+\varepsilon[) \cap X_1 : 0 \le p < 1,\ 0 < q < 1\} \cup \\ \cup \{(\{1/n\}\times]0, \varepsilon[) \cap X_1 : n \in \mathbb{N}\},\end{aligned}$$

and let $\{\mathfrak{c}_1(\varepsilon) : \varepsilon > 0\}$ form a base for the merotopy $\mathsf{M}_1$ on $X_1$. Both merotopies are compatible and Lodato; they are evidently accordant; the trace filters are Cauchy by the second line of (3). $\mathcal{U}(\mathsf{M}_0)$ and $\mathcal{U}(\mathsf{M}_1)$ have a common extension, namely the Euclidean uniformity on $X$. Let $\mathsf{M}$ be the Euclidean merotopy on $X$, which means that $\{\mathfrak{c}(\varepsilon) : \varepsilon > 0\}$ is a base for $\mathsf{M}$, where

$$\mathfrak{c}(\varepsilon) = \{(]p, p+\varepsilon[\times]q, q+\varepsilon[) \cap X : p, q \in \mathbb{R}\}.$$

Now $\Gamma(\mathsf{M})$ is an extension of $\Gamma(\mathsf{M}_0)$ and $\Gamma(\mathsf{M}_1)$. And yet, $\mathsf{M}_0$ and $\mathsf{M}_1$ cannot be extended, as 3.6 (1) is not fulfilled for $i = 1$, $j = 0$ and $\mathfrak{c}_i = \mathfrak{c}_1(1)$. $\diamondsuit$

## 4. Extending a family of contiguities in a closure space

### A. WITHOUT SEPARATION AXIOMS

**4.1** The exact counterparts of the results from §3 hold for contiguities. It is in fact possible to do the proofs all over again, inserting the word "finite" in appropriate places; it will be, however, simpler to deduce the results for contiguities from those for merotopies. We shall need some elementary (and well-known) facts about the connexion between contiguities and merotopies (the special case for $I = \emptyset$ of an extension problem to be discussed in Part IV):

Each contiguity $\Gamma$ can be induced by a coarsest merotopy $\mathsf{M}^0(\Gamma)$, for which $\Gamma$ (or any base for $\Gamma$) is a base; a merotopy of this form (i.e. one that has a base consisting of finite covers) is called *contigual.* $\Gamma$ is Riesz or Lodato iff $\mathsf{M}^0(\Gamma)$ has the same property. The function

$\Gamma \mapsto \mathsf{M}^0(\Gamma)$ gives a one-to-one correspondence between contiguities and contigual merotopies, keeps the relation finer/coarser, and commutes with the restriction to a subset as well as with taking the induced closure.

If a family of contiguities can be extended in a closure space then the closure is symmetric; similarly to the case of merotopies (and other structures), this condition is sufficient, too.

**Definitions.** For a family of contiguities in a closure space,

a) Let $\Gamma^0$ be the contiguity on $X$ for which the covers $\mathsf{f}_i^0$ $(i \in I, \mathsf{f}_i \in \Gamma_i)$ and $\mathsf{c}_{x,B}$ $(B \subset X, x \in c(B)^r)$ form a subbase.

b) Let $\Gamma^1$ consist of the finite covers $\mathsf{f}$ of $x$ that satisfy the following conditions: $\mathrm{St}(x, \mathsf{f}) \in \mathsf{v}(x)$ $(x \in X)$ and $\mathsf{f}|X_i \in \Gamma_i$ $(i \in I)$. $\diamondsuit$

In other words

$$(1) \qquad \Gamma^k = \Gamma(\mathsf{M}^k(c, \mathsf{M}^0(\Gamma_i))) \qquad (k = 0, 1).$$

**Theorem.** *A family of contiguities in a symmetric closure space always has extensions; $\Gamma^0$ is the coarsest and $\Gamma^1$ the finest extension.*

**Proof.** It follows from (1) and the foregoing observations that $\Gamma^0$ and $\Gamma^1$ are extensions. If $\Gamma$ is an extension then $\mathsf{M}^0(\Gamma)$ is an extension of the merotopies $\mathsf{M}^0(\Gamma_i)$, thus

$$\mathsf{M}^0(c, \mathsf{M}^0(\Gamma_i)) \subset \mathsf{M}^0(\Gamma) \subset \mathsf{M}^1(c, \mathsf{M}^0(\Gamma_i)),$$

implying $\Gamma^0 \subset \Gamma \subset \Gamma^1$. $\diamondsuit$

## B. RIESZ CONTIGUITIES IN A CLOSURE SPACE

**4.2** If a family of contiguities in a closure space has a Riesz extension then the contiguities are Riesz, the closure is weakly separated, and the trace filters are Cauchy. These conditions are also sufficient.

**Definition.** For a family of contiguities in a closure space, let

$$\Gamma^1_R = \{\mathsf{f} \in \Gamma^1 : \mathrm{int}\,\mathsf{f} \text{ is a cover of } X\}. \ \diamondsuit$$

This means that $\Gamma^1_R = \Gamma(\mathsf{M}^1_R(c, \mathsf{M}^0(\Gamma_i)))$.

**Theorem.** *A family of contiguities in a weakly separated closure space has a Riesz extension iff the trace filters are Cauchy; if so then $\Gamma^0$ is the coarsest and $\Gamma^1_R$ the finest Riesz extension.*

**Proof.** $\Gamma_i$-Cauchy means the same as $\mathsf{M}^0(\Gamma_i)$-Cauchy. $\diamondsuit$

## C. LODATO CONTIGUITIES IN A CLOSURE SPACE

**4.3** If a family of contiguities in a closure space has a Lodato extension then the contiguities are Lodato, the closure is an $S_1$-topology, and the trace filters are Cauchy. These conditions are not sufficient: modify again Example 1.8, or see 4.5 for a better example.

**Definitions.** For a family of Lodato contiguities in an $S_1$-space,

a) Let $\Gamma_L^1 = \{\mathsf{f} \in \Gamma^1 : \operatorname{int} \mathsf{f} \in \Gamma^1\}$.

b) Assuming that the trace filters are Cauchy, let $\Gamma_L^0$ be the contiguity on $X$ for which $\{\operatorname{int} \mathsf{f} : \mathsf{f} \in \Gamma^0\}$ is a base. ◊

Observe that $\Gamma_L^k = \Gamma(\mathsf{M}_L^k(c, \mathsf{M}^0(\Gamma_i)))$ $(k = 0, 1)$.

**Lemma.** *A family of Lodato contiguities in an $S_1$-space has a Lodato extension iff the trace filters are Cauchy and $\Gamma_L^0 \subset \Gamma_L^1$; if so then both $\Gamma_L^0$ and $\Gamma_L^1$ are Lodato extensions.* ◊

**Theorem.** *A family of Lodato contiguities in an $S_1$-space has a Lodato extension iff the trace filters are Cauchy, and, for any $i, j \in I$,*

$$(1) \qquad (\operatorname{int} \mathsf{f}_i^0)|X_j \in \Gamma_j \qquad (\mathsf{f}_i \in \Gamma_i);$$

*if so then $\Gamma_L^0$ is the coarsest and $\Gamma_L^1$ the finest Lodato extension.* ◊

**Corollary.** *A family of contiguities in an $S_1$-space has a Lodato extension iff $\{\Gamma_i, \Gamma_j\}$ has a Lodato extension for any $i, j \in I$.* ◊

**4.4 Corollary.** *A single Lodato contiguity given in an $S_1$-space has a Lodato extension iff the trace filters are Cauchy.* ◊

**4.5 Theorem.** *Let a family of Lodato contiguities be given in an $S_1$-space, assume that the trace filters are Cauchy, and 3.8 (1) holds. Then there exists a Lodato extension.* ◊

**Corollary.** *Let a family of Lodato contiguities be given in an $S_1$-space. Assume that either each $X_i$ is open and the trace filters are Cauchy or each $X_i$ is closed. Then there exists a Lodato extension.* ◊

**Example.** Let $X, X_0, X_1, c$ and $\mathsf{M}_0$ be as in Example 3.8. Take $\Gamma_0 = \Gamma(\mathsf{M}_0)$, and let $\{\mathsf{f}_1(k) : k \in \mathbb{N}\}$ be a subbase for $\Gamma_1$ on $X_1$, where

$$\begin{aligned} \mathsf{f}_1(k) = \{&\{(1/m, 1/n) : m, n \geq k, m \not\equiv (\operatorname{mod} 3)\} : \mu = 0, 1, 2\} \cup \\ \cup &\{\{(\{1/m, 1/n) : n \geq k\} : m < k\} \cup \\ \cup &\{\{(1/m, 1/n) : m \geq k\} : n < k\} \cup \\ \cup &\{\{(1/m, 1/n)\} : m, n < k\}. \end{aligned}$$

Now $\{\Gamma_0, \Gamma_1\}$ is a family of Lodato contiguities, the trace filters are

Cauchy, the induced proximities have a Lodato extension (the Euclidean one on $X$), but $\Gamma_0$ and $\Gamma_1$ do not have one, as 4.3 (1) fails for $i = 1$, $j = 0$, $\mathsf{f}_i = \mathsf{f}_1(1)$. ◇

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