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## NOTE ON MODULES OVER PRÜFER DOMAINS\*

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**Abstract:** We give various characterizations – in terms of module properties – for Prüfer domains in general, and for (locally) almost maximal Prüfer domains, in particular. A domain  $R$  is a Prüfer domain if and only if pure-injective divisible  $R$ -modules are injective. A Prüfer domain  $R$  is locally almost maximal exactly if finitely embedded  $R$ -modules are pure-injective. An  $h$ -local domain  $R$  is almost maximal Prüfer if and only if finitely embedded  $R$ -modules are direct sums of cocyclic  $R$ -modules.

All rings will be commutative with 1. A ring  $R$  is *maximal* if it is linearly compact in the discrete topology (this is the topology in which

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linear compactness will be considered here);  $R$  is *almost maximal* if the ring  $R/I$  is maximal for every non-zero ideal  $I$  of  $R$ . A domain  $R$  (a ring without divisors of zero) is a *valuation domain* if its ideals form a chain under inclusion; it is a *Prüfer domain* if its finitely generated ideals are projective, or, equivalently, its localization  $R_M$  is a valuation domain for every maximal ideal  $M$  of  $R$ . If all of these localizations  $R_M$  are almost maximal,  $R$  is said to be *locally almost maximal* (Brandal [2, p.20]). A domain  $R$  is called  *$h$ -local* if (i) every non-zero prime ideal of  $R$  is contained in exactly one maximal ideal, and (ii) every non-zero element is contained in but a finite number of maximal ideals. By Brandal [2, Th. 2.9], a domain is almost maximal if and only if it is an  $h$ -local, locally almost maximal domain.

A module  $D$  over a domain  $R$  is called *divisible* if  $rD = \{rd \mid d \in D\}$  is equal to  $D$  for all  $0 \neq r \in R$ , and  *$h$ -divisible* if it is an epic image of a direct sum of copies of the field  $Q$  of quotients of  $R$  (as an  $R$ -module).  $D$  is *absolutely pure* if it is pure in every  $R$ -module in which it is contained. Megibben [8] has shown (in a more general form) that a domain  $R$  is Prüfer if and only if every divisible module is absolutely pure. Naudé-Naudé-Pretorius [9] proved that a domain  $R$  is Prüfer exactly if all pure-injective modules are  $RD$ -injective ( $RD$ -injectivity is defined as the injective property relative to inclusions  $A \rightarrow B$  where  $rA = A \cap rB$  for all  $r \in R$ ; see [5, p.210]). This result will be sharpened: it is enough to require that the pure-injective divisible modules be  $RD$ -injective. See Theorem 5.

An  $R$ -module  $C$  is said to be *cocyclic* if it is an essential extension of a simple  $R$ -module  $S$ , i.e. it is contained in the injective hull  $E(S)$  of  $S$ . An  $R$ -module  $F$  is *finitely embedded* if it is an essential extension of a finite direct sum of simple  $R$ -modules. In investigating classical rings  $R$  (i.e.  $E(S)$  is linearly compact for every simple  $R$ -module  $S$ ), Vámos [10] identified the classical Prüfer domains as those classical domains over which the finitely embedded modules are direct sums of cocyclic modules. We prove a similar result (Theorem 8) characterizing the  $h$ -local domains over which such decompositions hold: these are exactly the almost maximal Prüfer domains. This is the dual of a result by Matlis [7, Th. 5.7] which deals with the decompositions of finitely generated modules into direct sums of cyclics. Those Prüfer domains will also be described over which the finitely embedded modules are linearly compact (or pure-injective); see Theorem 6. A similar problem

was investigated by Facchini [4]: he characterized the rings over which finitely embedded modules have injective dimension  $\leq 1$ . (We wish to thank Willy Brandal for calling our attention to this paper.)

For unexplained terminology we refer to standard texts or to Fuchs-Salce [5].

## 1. Preliminaries

We start our discussion with lemmas on modules over arbitrary domains  $R$ . For an  $R$ -module  $D$  and  $r \in R$ , we set  $D[r] = \{d \in D \mid rd = 0\}$ .

**Lemma 1.** *The  $R$ -module  $\text{Hom}_R(D, *)$  is torsion-free whenever  $D$  is a divisible  $R$ -module.*

**Proof.** From the exact sequence  $0 \rightarrow D[r] \rightarrow D \xrightarrow{r} D \rightarrow 0$  we infer that the sequence  $0 \rightarrow \text{Hom}(D, *) \xrightarrow{r} \text{Hom}(D, *) \rightarrow \text{Hom}(D[r], *)$  is exact.  $\diamond$

**Lemma 2.** *If  $A$  is a torsion-free and  $E$  is an injective  $R$ -module, then  $\text{Hom}_R(A, E)$  is divisible and pure-injective.*

**Proof.** The pure-injectivity of  $\text{Hom}_R(*, E)$  for  $E$  pure-injective is well known (see e.g. [5, p.217]). The exact sequence  $0 \rightarrow E[r] \rightarrow E \xrightarrow{r} E \rightarrow 0$  implies the exactness of

$$0 \rightarrow \text{Hom}(A, E[r]) \rightarrow \text{Hom}(A, E) \xrightarrow{r} \text{Hom}(A, E) \rightarrow \text{Ext}^1(A, E[r]).$$

As  $E[r]$  is  $RD$ -injective (see [5, p.210]) and  $A$  is torsion-free, the last term vanishes. Hence  $\text{Hom}(A, E)$  is divisible.  $\diamond$

**Lemma 3.** *The pure-injective hull of a divisible module is divisible.*

**Proof.** If  $E$  is an injective cogenerator of the category of  $R$ -modules, then for every  $R$ -module  $M$ , there is a pure embedding

$$M \rightarrow \text{Hom}_R(\text{Hom}_R(M, E), E) = H$$

and the pure-injective hull  $PE(M)$  of  $M$  is a summand of the pure-injective module  $H$  (see [5, p.217]). It is therefore enough to show that if  $M$  is divisible, then so is  $H$ . By Lemma 1, if  $M$  is divisible, then  $\text{Hom}(M, E)$  is torsion-free. Hence Lemma 2 implies  $H$  is divisible.  $\diamond$

**Lemma 4.** *An  $RD$ -injective divisible module is injective.*

**Proof.** By [5, p.213], an  $RD$ -injective module  $M$  decomposes as  $M = E \oplus N$  where  $E$  is injective and  $N^1 = \bigcap_{0 \neq r \in R} rN = 0$ . If  $M$  is divisible,

then necessarily  $N = 0$ , and  $M = E$  is injective.  $\diamond$

**Remark.** Actually, the following converse of Lemma 2 holds: *if  $E$  is an injective cogenerator of the category of  $R$ -modules, then  $\text{Hom}_R(A, E)$  is (pure-injective) divisible if and only if  $A$  is torsion-free.* To see this, consider the isomorphism [3, p.120]

$$\text{Ext}_R^1(B, \text{Hom}_R(A, E)) \cong \text{Hom}_R(\text{Tor}_1^R(B, A), E)$$

which holds for all  $R$ -modules  $A, B$  and injective  $E$ . Recall that an  $R$ -module  $D$  is divisible exactly if  $\text{Ext}_R^1(R/Rr, D) = 0$  for all  $r \in R$  [5, p.36]. In view of the above isomorphism,  $D = \text{Hom}_R(A, E)$  is divisible if and only if, for all  $r \in R$ ,  $\text{Hom}_R(\text{Tor}_1^R(R/Rr, A), E) = 0$ . This amounts to  $\text{Tor}_1^R(R/Rr, A) = 0$  whenever  $E$  is an injective cogenerator. The exact sequence  $0 \rightarrow R \xrightarrow{r} R \rightarrow R/Rr \rightarrow 0$  induces the exact sequence  $0 \rightarrow \text{Tor}_1(R/Rr, A) \rightarrow R \otimes A \cong A \xrightarrow{r} R \otimes A \cong A$ . This shows that  $\text{Tor}_1(R/Rr, A) = 0$  for all  $r \in R$  is equivalent to the torsion-freeness of  $A$ .

The reader is advised to compare our remark with the well-known fact that if  $E$  is an injective cogenerator, then the injectivity of  $\text{Hom}_R(A, E)$  is equivalent to the flatness of  $A$ . (Hence the equivalence of (i) and (ii) in Theorem 5 can easily be derived: just recall flatness and torsion-freeness are equivalent exactly for Prüfer domains.)

## 2. Characterizations of Prüfer domains

The next result gives various equivalent properties which characterize Prüfer domains among the domains. The equivalence of (i) and (iv) is due to Megibben [8], while the equivalence of (i) and (ii) improves on a result by Naudé-Naudé-Pretorius [9].

**Theorem 5.** *For a domain  $R$ , the following are equivalent:*

- (i)  $R$  is a Prüfer domain;
- (ii) pure-injective divisible  $R$ -modules are injective;
- (iii) pure-injective hulls of divisible  $R$ -modules are injective;
- (iv) divisible  $R$ -modules are absolutely pure;
- (v)  $h$ -divisible  $R$ -modules are absolutely pure.

**Proof.** (i)  $\Rightarrow$  (ii): For Prüfer domains, purity and  $RD$ -property are equivalent (see [5, p.47]). Hence Lemma 4 shows that (ii) holds for Prüfer domains.

(ii)  $\Rightarrow$  (iii) is obvious in view of Lemma 3.

(iii)  $\Rightarrow$  (iv): Let  $D$  be a divisible module in the exact sequence  $0 \rightarrow D \rightarrow A \rightarrow B \rightarrow 0$ . Using the canonical embedding  $\delta : D \rightarrow PE(D)$ , form the pushout diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & D & \rightarrow & A & \rightarrow & B \rightarrow 0 \\ & & \downarrow \delta & & \downarrow \alpha & & \parallel \\ 0 & \rightarrow & PE(D) & \xrightarrow{\gamma} & C & \rightarrow & B \rightarrow 0 \end{array}$$

where  $\alpha$  is monic. By (iii),  $PE(D)$  is injective, and therefore  $\text{Im } \gamma$  is a summand of  $C$ . It follows that  $\gamma\delta D$  is pure in  $C$ , and so  $D$  is pure in  $A$ . Thus  $D$  is absolutely pure.

(iv)  $\Rightarrow$  (v) is trivial.

(v)  $\Rightarrow$  (i): Let  $L$  be a finitely generated ideal of  $R$ , and  $D$  an  $h$ -divisible  $R$ -module. By (v),  $D$  is absolutely pure, thus every extension of  $D$  by a finitely presented  $R$ -module is splitting. In particular,  $\text{Ext}^1(R/L, D) = 0$ . Given an  $R$ -module  $M$  and its injective hull  $E$ , the module  $D$  in the exact sequence  $0 \rightarrow M \rightarrow E \rightarrow D \rightarrow 0$  is  $h$ -divisible. Form the commutative diagram

$$\begin{array}{ccccccc} \text{Hom}(R, E) & \rightarrow & \text{Hom}(R, D) & \rightarrow & 0 & & \\ \downarrow & & \downarrow & & & & \\ \text{Hom}(L, E) & \rightarrow & \text{Hom}(L, D) & \rightarrow & \text{Ext}^1(L, M) & \rightarrow & 0 \\ \downarrow & & \downarrow & & & & \\ \text{Ext}^1(R/L, E) = 0 & & \text{Ext}^1(R/L, D) = 0 & & & & \end{array}$$

with exact rows and columns. The composite map  $\text{Hom}(R, E) \rightarrow \text{Hom}(L, D)$  being surjective,  $\text{Ext}^1(L, M) = 0$  follows. This holds for every  $M$ , so  $L$  is projective and  $R$  is Prüfer.  $\diamond$

### 3. Locally almost maximal Prüfer domains

Among the valuation domains, the almost maximal ones are distinguished by a number of attractive properties. Some of these properties carry over to almost maximal Prüfer domains. We are particularly interested in those which relate to the finitely embedded modules.

**Theorem 6.** *For a Prüfer domain  $R$ , the following are equivalent:*

- (i)  $R$  is locally almost maximal;
- (ii) finitely embedded  $R$ -modules are linearly compact;
- (iii) finitely embedded  $R$ -modules are pure-injective;
- (iv) cocyclic  $R$ -modules are pure-injective.

**Proof.** (i)  $\Rightarrow$  (ii): If  $R_M$  is an almost maximal valuation domain for every maximal ideal  $M$ , then  $Q/R_M M$  is linearly compact for every  $M$ , both as an  $R_M$ - and as an  $R$ -module. A finitely embedded  $R$ -module is a submodule of a finite direct sum of linearly compact  $R$ -modules of the form  $Q/R_M M$ , hence itself linearly compact.

(ii)  $\Rightarrow$  (iii) is clear, since linear compactness over a commutative ring always implies pure-injectivity.

(iii)  $\Rightarrow$  (iv) is trivial.

(iv)  $\Rightarrow$  (i): The  $R$ -module  $Q/R_M M$  is cocyclic, and therefore pure-injective. It is moreover, injective, since over Prüfer domains divisible pure-injective modules are injective (cf. Theorem 5). The exact sequence  $0 \rightarrow R_M/R_M M \rightarrow Q/R_M M \rightarrow Q/R_M \rightarrow 0$  implies the exactness of

$$0 = \text{Ext}_R^1(R/I, Q/R_M M) \rightarrow \text{Ext}_R^1(R/I, Q/R_M) \rightarrow \\ \rightarrow \text{Ext}_R^2(R/I, R_M/R_M M)$$

for every ideal  $I$  for  $R$ . Simple modules are always  $RD$ -injective, and hence they have injective dimension 1 [5, p.243]. If the last Ext vanishes, then so does the middle one. This implies that  $Q/R_M$  is (an injective  $R$ -module and so) an injective  $R_M$ -module, proving the almost maximality of  $R_M$ .  $\diamond$

#### 4. $h$ -local almost maximal Prüfer domains

Our final goal is to find all  $h$ -local domains over which the finitely embedded modules are direct sums of cocyclics.

Recall that a torsion module  $T$  over an  $h$ -local domain  $R$  is the direct sum of its localizations:  $T_M = R_M \otimes_R T$ . Here  $T_M$  is an  $R_M$ -module whose  $R$ - and  $R_M$ -module structures coincide (see Brandal [1, Lemma 2.7]).

We start with a lemma; this is the dual of a result by Matlis [7] and Gill [6].

**Lemma 7.** *Let  $R$  be a local domain. If every finitely embedded  $R$ -*

module is a direct sum of cocyclic  $R$ -modules, then  $R$  is a valuation domain.

**Proof.** Suppose  $R$  has the stated property, but is not a valuation domain. Choose  $a, b \in R$  such that  $b \notin Ra$  and  $a \notin Rb$ . There is an ideal  $A$  of  $R$  which is maximal with respect to the properties  $a \in A$  and  $b \notin A$ . Similarly, there is an ideal  $B$  of  $R$  maximal with respect to  $b \in B$ ,  $a \notin B$ . Consider the  $R$ -module  $F = R/(A \cap B)$  which is evidently a submodule of  $R/A \oplus R/B$ . Here  $R/A$  is subdirectly irreducible with  $b + A$  generating its socle; thus  $R/A$  is cocyclic. The same holds for  $R/B$ . We conclude that  $R/A \oplus R/B$  and hence  $F$  is finitely embedded. Neither  $a$  nor  $b$  is a unit of  $R$ , thus both  $A$  and  $B$  are contained in the maximal ideal  $M$  of  $R$ . Consequently,  $F$  is indecomposable, and hence – by hypothesis – cocyclic. But  $F$  has a non-simple socle  $R(b + A) \oplus R(a + B)$ , a contradiction.  $\diamond$

Observe that the last lemma holds for all commutative local rings.

We are now able to prove the dual of a theorem of Matlis [7, Th. 5.7]. (Since the ring is not assumed to be classical, duality arguments can not be applied.)

**Theorem 8.** *Let  $R$  be an  $h$ -local domain. The following are equivalent:*

- (a)  *$R$  is an almost maximal Prüfer domain;*
- (b) *every finitely embedded  $R$ -module is a direct sum of cocyclic  $R$ -modules.*

**Proof.** (a)  $\Rightarrow$  (b): Since  $R$  is  $h$ -local, every finitely embedded  $R$ -module  $F$  is a finite direct sum  $F = \oplus F_M$  where  $F_M$  is a finitely embedded  $R_M$ -module. The  $R$ - and  $R_M$ -module structures of  $F_M$  are identical, thus it suffices to verify the implication for an almost maximal valuation domain  $R$  (with maximal ideal  $M$ ).

In this case, the injective hull  $E$  of a finitely embedded  $R$ -module  $F$  is the direct sum of a finite number of copies of  $Q/M$ . Hence we conclude that  $F$  is a submodule of a finite direct sum of uniserial  $R$ -modules, and so it is polyserial in the sense of [5,p.190]. Polyserial torsion modules over an almost maximal valuation domain are direct sums of uniserials, hence (b) holds.

(b)  $\Rightarrow$  (a): We argue as before that it is enough to prove that a local domain  $R$  with property (b) has to be an almost maximal valuation domain.

That  $R$  is a valuation domain has been proved in Lemma 7. By way of contradiction, suppose  $R$  is not almost maximal. Then there is

a unit  $u$  in a maximal immediate extension  $\widehat{R}$  of  $R$  which is not in  $R$  and whose breadth ideal

$$B = B(u) = \{r \in R | u \notin R + r\widehat{R}\} \neq 0.$$

For every  $x \in R \setminus B$  there is a (unit)  $u_x \in R$  such that  $u - u_x \in Rx$ . Visibly, the family of units  $\{u_x \in R | x \in R \setminus B\}$  satisfies

- (i)  $u_x - u_y \in Rx$  if  $y \in Rx$ ,
- (ii) there is no  $v \in R$  such that  $v - u_x \in Rx$  for all  $x \in R \setminus B$ .

Define the fractional ideal  $C = B^{-1} = \{q \in Q | qB \leq R\}$ ; thus for  $r \in R$ ,  $r^{-1} \in C$  exactly if  $B \leq rR$ , i.e.

$$C = \bigcup_{r \in R \setminus B} Rr^{-1}.$$

Multiplication by  $u_x$  induces an automorphism  $\alpha_x$  of  $C/R$ . If  $y \in Rx$ , then  $u_x x^{-1} - u_y x^{-1} \in R$  shows that  $\alpha_x w = \alpha_y w$  for all  $w \in Rx^{-1}$ . The automorphism  $\alpha$  of  $C/R$  defined by  $\alpha w = \alpha_x w$  for  $w \in Rx^{-1}$  is not induced by any element of  $R$ .

For some non-unit  $t$  of  $R$ , consider the cocyclic uniserial  $R$ -module  $V = C/Mt$ . There is no automorphism  $\theta$  of  $U$  which would induce  $\alpha$  on  $C/R$ , because of the choice of  $C$ . Using the submodule  $V = R/Mt$  and the canonical map  $\pi : U \rightarrow U/V$ , form the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 & & & & V & = & V \\
 & & & & \downarrow & & \downarrow \\
 0 & \rightarrow & V & \rightarrow & X & \rightarrow & U \rightarrow 0 \\
 & & \parallel & & \downarrow & \nearrow & \downarrow \alpha\pi \\
 0 & \rightarrow & V & \rightarrow & U & \xrightarrow{\pi} & U/V \rightarrow 0
 \end{array}$$

Since there is no automorphism  $\theta : U \rightarrow U$  making the arising lower triangle commute in either direction, neither the middle row nor the middle column splits. Manifestly,  $V \oplus V \leq X \leq U \oplus U$ , so  $X$  is finitely embedded, and as such it is a direct sum  $X = X_1 \oplus X_2$  where  $X_i$  are cocyclic. The proof of [5, p.190] shows that the intersection of  $X$  with one of the  $U$ 's is pure in  $X$ . This amounts to the purity of one of



the  $V$ 's in  $X$ . By [5, p.192], pure submodules of a finite direct sum of uniserials are summands; consequently, either the middle row or the middle column splits. This contradiction shows that no  $u \in \widehat{R}$  can exist with  $B(u) \neq 0$ , i.e.  $R$  is almost maximal.  $\diamond$

The characterization of rings  $R$  for which part (b) of Theorem 8 holds is an open question. The condition of  $R$  being  $h$ -local can be weakened by demanding only that every prime  $\neq 0$  in  $R$  be contained in exactly one maximal ideal of  $R$ .

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