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DISTRIBUTION AND MOMENT CONVERGENCE OF SUMS OF ASSOCIATED RANDOM VARIABLES

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Abstract: Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with zero mean, and let $S_{n,k} = \sum_{1 \leq i_1 < \ldots < i_k \leq n} X_{i_1} \ldots X_{i_k}$, $k \geq 1$. We present sufficient conditions for the distribution and moment convergence of $S_{n,k}/\mathrm{Var}(S_{n,1})$, to the distribution and moments of $H_k(\mathcal{N})/k!$, where H_k is the Hermite polynomial of degree k and $\mathcal{N}$ is a standard normal variable.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables, defined on some probability space $(\Omega, \Im, P)$, such that $EX_n = 0$, $EX_n^2 < \infty$, $n \geq 1$.

Let us put:

$$S_0 = 0,\ S_n = \sum_{k=1}^{n} X_k,\ \sigma_n^2 = ES_n^2,\ n \geq 1,$$

$$S_{n,k} = \sum_{1 \leq i_1 < \ldots < i_k \leq n} X_{i_1} X_{i_2} \ldots X_{i_k},\quad k \geq 1$$

$$Y_{n,k} = S_{n,k}/\sigma_n,\ U_n^2 = \sum_{i=1}^{n} X_i^2/\sigma_n^2.$$

Let us observe that $S_{n,1} = S_n$.

Let $H_k(z)$ denote the Hermite polynomial of degree k, defined by

$$\left(\frac{d}{dz}\right) \exp(-z^2/2) = (-1)^k H_k(z) \exp(-z^2/2).$$

One can note that setting $H_0(z) \equiv 1$ we have

$$H_{k+1} = zH_k(z) - kH_{k-1}(z),\quad k \geq 1.$$

Let $\mathcal{N}$ denote a standard normal variable.

In this paper we study convergence of distributions and moments of the sequence $\{Y_{n,k}, n \geq 1\}$ to the distribution and moments of $H_k(\mathcal{N})/k!$. This problem has been studied by Teicher [5] in the case when $\{X_n, n \geq 1\}$ is a sequence of square-integrable martingale differences. We investigate sequences $\{X_n, n \geq 1\}$ that satisfy a condition of positive dependence called association.

We recall that a collection $\{X_1, \ldots, X_n\}$ of random variables is *associated* if for any two coordinatewise nondecreasing functions f_1, f_2 on $\mathbb{R}^n$ such that $\widehat{f_i} = f_i(X_1, \ldots, X_n)$ has finite variance for $i = 1, 2$, thus $\mathrm{Cov}(\widehat{f_1}, \widehat{f_2}) \geq 0$ holds. An infinite collection is associated if every finite subcollection is associated (cf. [4]).

Recently many papers have been published concerning weak convergence of the sequence $\{Y_{n,1}, n \geq 1\}$ to the distribution of $H_1(\mathcal{N})$ (cf. [2] and the references given there). But, as we know, there are no results concerning weak convergence of the sequence $(Y_{n,k}, n \geq 1)$ for associated sequences. The convergence of moments of $\{Y_{n,k}, n \geq 1\}$ has not been studied yet, even in the case $k = 1$. The results presented in this paper fill in this gap.

2. Results

Studying limit properties of associated sequences we need the following coefficient

$$u(n) = \sup_{n \in \mathbb{N}} \sum_{j:|j-k| \geq n} \operatorname{Cov}(X_j, X_k), \quad n \in \mathbb{N} \cup \{0\}.$$

In what follows we will use the following conditions

(1) $\lim_{n\to\infty} E(U_n^2)^p = 1$,
(2) $\sup_{n\in\mathbb{N}} E|X_n|^{2p+\delta} = M < \infty$,
(3) $u(n) = O(n^{-(2p+\delta/2-2)(2p+\delta)/\delta})$, and $u(1) < \infty$,
(4) $\inf_{n\in\mathbb{N}} n^{-1}\sigma_n^2 > 0$.

Theorem 1. *Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables such that $EX_n = 0$ and $EX_n^2 < \infty$, $n \geq 1$. If for some $p > 1$ and $\delta \in (0,1)$ the conditions (2), (3) and (4) hold, then*

$$S_n/\sigma_n \xrightarrow{D} \mathcal{N} \quad \text{as} \quad n \to \infty, \tag{5}$$

and

$$E|S_n/\sigma_n|^{2p} \longrightarrow E|\mathcal{N}|^{2p} \quad \text{as} \quad n \to \infty. \tag{6}$$

Theorem 2. *Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables such that $EX_n = 0$ and $EX_n^2 < \infty$, $n \geq 1$. If for some $p > 1$ and $\delta \in (0,1)$ the conditions (1), (2), (3) and (4) hold then for every $k \in \mathbb{N}$*

$$Y_{n,k} \xrightarrow{D} H_k(\mathcal{N})/k! \quad \text{as} \quad n \to \infty, \tag{7}$$

and

$$(8) \qquad E|Y_{n,k}|^{2p/k} \longrightarrow E|H_k(\mathcal{N})/k!|^{2p/k} \quad as \quad n \to \infty.$$

3. Proofs

Proof to Theorem 1. By (4), $\sigma_n^2 \to \infty$ as $n \to \infty$. Thus, by (2) and (4), we get the Lyapunov condition of order $2p+\delta$

$$\sigma_n^{-2p-\delta} \sum_{i=1}^{n} E|X_i|^{2p+\delta} \leq nM\sigma^{-2p-\delta} =$$

$$= M(n/\sigma_n^2)\sigma^{-2(p-1)-\delta} \to 0 \quad as \quad n \to \infty.$$

Hence the Lindeberg condition of order $2p+\delta$ holds and therefore the classical Lindeberg condition is satisfied:

$$(9) \quad \sigma_n^{-2} \sum_{i=1}^{n} E|X_i|^2 I[|X_i| \geq \sigma_n \varepsilon] \to 0 \text{ as } n \to \infty \text{ for every } \varepsilon > 0.$$

Taking into account (3), (4) and (9), by Theorem 3 of Birkel [2] we have (5).

By (4) we have $n/\sigma^2 < C < \infty$ for some constant C and every $n \in \mathrm{N}$. Thus applying Theorem 1 of Birkel [3] we get

$$E|S_n/\sigma_n|^{2p+\delta/2} \leq \sup_{n \in \mathbb{N} \cup \{0\}} E|S_{n+m} - S_m|^{2p+\delta/2}/\sigma_n^{2p+\delta/2} \leq$$

$$\leq Bn^{(2p+\delta/2)/2}/(\sigma_n^2)^{(2p+\delta/2)/2} \leq K,$$

where K is a constant not depending on n.

Thus the sequence $\{|S_n/\sigma_n|^{2p}, n \geq 1\}$ is uniformly integrable so, by Theorem 5.4 of Billingsley [1] we get (6).

Proof of Theorem 2. In order to obtain (7) it suffices, by Remark 1 of Teicher [5], to prove the following three assertions:

(i) $S_n/\sigma_n \xrightarrow{D} \mathcal{N}$,

(ii) $U_n^2 \xrightarrow{P} 1$,

(iii) $\max_{1\le i\le n} |X_i|/\sigma_n \xrightarrow{P} 0$ as $n \to \infty$.

(i) follows from Theorem 1.
To prove (ii), we first show that

$$EU_n^2 \to 1 \quad \text{as} \quad n \to \infty. \tag{10}$$

Clearly:

$$1 \ge EU_n^2 = \sigma_n^{-2} E(X_1^2 + \ldots + X_n^2) = 1 - \sigma_n^{-2} \sum_{1\le i\ne j\le n}\sum \operatorname{Cov}(X_i, X_j) \ge$$

$$\ge 1 - 2\sigma_n^{-2} \sum_{j=1}^{n} u(j) \ge 1 - 2\sigma_n^{-2} \sum_{j=1}^{\infty} u(j).$$

But by assumption (3), for sufficiently large j we have

$$u(j) \le C/j^{(1+\delta/4)},$$

thus $\sum_{j=1}^{\infty} u(j) < \infty$, and therefore

$$1 \ge EU_n^2 \ge 1 - 2\sigma_n^{-2} \textstyle\sum_{j=1}^{\infty} u(j) \to 1 \quad \text{as} \quad n \to \infty.$$

By (1) and (10), applying Lemma 1 of Teicher [5] we get

$$E|U_n^2 - 1|^p \to 0 \text{ as } n \to \infty, \text{ thus } U_n^2 \xrightarrow{P} 1 \text{ as } n \to \infty.$$

To prove (iii) let us observe that under our assumptions the Lyapunov conditon holds and therefore

$$P[\max_{1\le i\le n} |X_i|/\sigma_n \ge \varepsilon] \le \sum_{i=1}^{n} P[|X_i|/\sigma_n \ge \varepsilon] \le$$

$$\le \varepsilon^{-2p-\delta} \sigma_n^{-2p-\delta} \sum_{i=1}^{n} E|X_i|^{2p+\delta} \to 0 \text{ as } n \to \infty.$$

Thus (i), (ii) and (iii) hold hence (7) is satisfied.

By Theorem 1 we have (8) in the case $k = 1$. Taking into account condition (1) and applying Lemma 1 of Teicher [5] we get the uniform integrability of the sequence $\{U_n^{2p}, n \geq 1\}$. From the proof of Theorem 1 the sequence $\{|Y_n|^{2p}, n \geq 1\}$ is uniformly integrable, where $Y_n = Y_{n,1}$. Let us observe that

$$|2Y_{n,2}|^p = |Y_n^2 - U_n^2|^p \leq 2^p(|Y_n|^{2p} + U_n^{2p}),$$

thus the sequence $\{|Y_{n,2}|^p, \ n \geq 1\}$ is uniformly integrable. This fact and (7) yields (8) in the case $k = 2$.

Now we proceed by induction on k.
Assume that (8) holds for $m = 1, 2, \ldots, k$ then by (7) and Theorem 5.4 of Billingsley [1], the sequence $\{|Y_{n,m}|^{2p/m}, n \geq 1\}$ is uniformly integrable for $m = 1, 2, \ldots, k$. For $0 \leq j < k$ we have

$$|Y_{n,k-j} \sum_{i=1}^{n} (X_i/\sigma_n)^{j+1}| \leq (k-j)/(k+1)|Y_{n,k-j}|^{(k+1)/(k-j)} +$$

$$+(j+1)/(k+1)| \sum_{i=1}^{n} (X_i/\sigma_n)^{j+1}|^{(k+1)/(j+1)},$$

and

$$|Y_{n,k-j} \sum_{i=1}^{n} (X_i/\sigma_n)^{j+1}|^{2p/(k+1)} \leq C \left(\frac{k-j}{k+1}\right)^{2p/(k+1)} |Y_{n,k-j}|^{2p/(k-j)} +$$

$$+ C \left(\frac{j+1}{k+1}\right)^{2p/(k+1)} | \sum_{i=1}^{n} (X_i/\sigma_n)^{j+1}|^{2p/(j+1)} \leq$$

$$\leq C \left(|Y_{n,k-j}|^{2p/(k-j)} + | \sum_{i=1}^{n} (X_i/\sigma_n)^{j+1}|^{2p/(j+1)} \right),$$

where $C = 2^{2p/(k+1)}$.
Let us observe that for $1 \leq j < k$ we have

$$| \sum_{i=1}^{n} X_i^{j+1}|^{2p/(j+1)} \leq \left(\sum_{i=1}^{n} X_i^2 \right)^p.$$

Thus taking into account the uniform integrability of $\{U_n^{2p}, n \geq 1\}$ and $\{|Y_{n,m}|^{2p/m}, n \geq 1\}$, for $m = 1, 2, \ldots, k$, and the above inequalities we see that the sequence

$$\{|Y_{n,k-j} \sum_{i=1}^{n} (X_i/\sigma_n)^{j+1}|^{2p/(k+1)}, n \geq 1\}$$

is uniformly integrable for $0 \leq j \leq k$.
Now using the equality

$$(k+1)Y_{n,k+1} = \sum_{j=0}^{k} (-1)^j Y_{n,k-j} \left(\sum_{i=1}^{n} (X_i/\sigma_n)^{j+1} \right)$$

we get the uniform integrability of the sequence

$$\left\{ |Y_{n,k+1}|^{2p/(k+1)}, n \geq 1 \right\},$$

this together with (7) implies (8) in the case $m = k+1$, which completes the proof of Theorem 2.

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