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A NON-COMPLETELY REGULAR QUIET QUASI-METRIC*

J. Deák

Mathematical Institute of the Hungarian Academy of Sciences, H-1053 Budapest, Reáltanoda u. 13 - 15, Hungary.

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Abstract: We improve on an example of Fletcher, Hejcman and Hunsaker [4] for a non-completely regular quiet-uniformity: our example is quasi-metrizable.

1. Introduction

Doitchinov [2,3] introduced a class of quasi-metrics, respectively quasi-uniformities, admitting a satisfactory theory of completeness and completion. We shall only deal here with these classes, and not with completions. See [5] for basic definitions concerning quasi-metrics and quasi-uniformities.

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Definitions. A *sequence pair* (a *filter pair*) is an ordered pair of sequences (of filters).

The sequence pair $(< x_k >, < y_n >)$ in the quasi-metric space (X, d) is *Chauchy* if for any $\varepsilon > 0$, there is an $m \in \mathbb{N}$ such that $d(x_k, y_n) < \varepsilon \quad (k, n > m)$. A filter pair (f,g) in the quasi-uniform space $(X, \mathcal{U})$ is *Chauchy* if for any $U \in \mathcal{U}$, there are $F \in \mathsf{f}$ and $G \in \mathsf{g}$ with $F \times G \subset U$.

The quasi-metric d is *balanced* [2] if for any Chauchy sequence pair $(< x_k >, < y_n >)$, and for any $x, y \in X$, we have

$$(1) \qquad d(x,y) \le \sup_n d(x, y_n) + \sup_k d(x_k, y).$$

(Equivalently: one can write lim sup instead of sup.) The T_1-quasi-uniformity $\mathcal{U}$ is *quiet* [3] provided that for any $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that if $x, y \in X$, (f, g) is a Cauchy filter pair, $Vx \in \mathsf{g}, V^{-1}y \in \mathsf{f}$, then xUy. (See [1] §§7 – 8 for related notions.) ◊

The notions of a balanced quasi-metric and of a quiet quasi-uniformity are in close connexion: if d is balanced then $\mathcal{U}(d)$ is quiet ([3] p. 6); it is also pointed out in [3] that the quietness of $\mathcal{U}(d)$ can be reformulated in terms of d, namely: $\mathcal{U}(d)$ is quiet iff for any $\varepsilon > 0$ there is a $\delta > 0$ such that $d(x,y) < \varepsilon$ whenever there is a Cauchy sequence pair $(< x_k >, < y_n >)$ with $d(x, y_n) < \delta \quad (n \in \mathbb{N})$ and $d(x_k, y) < \delta$ $(k \in \mathbb{N})$.

Conversely, if $\mathcal{U}$ is quasi-metrizable and quiet then it can be induced by a quasi-metric d satisfying a condition strictly stronger than the one in the preceding paragraph, but strictly weaker than the one in the definition of balanced quasi-metrics: there is a constant C such that for any Cauchy sequence pair $(< x_k >, < y_n >)$, and for any $x, y \in X$,

$$(2) \qquad d(x,y) \le C(\sup_n d(x, y_n) + \sup_k d(x_k, y)).$$

A routine application of the Metrization Lemma ([6], 6.12) gives this with $C = 8$; taking then $d' = \sqrt[j]{d}$ with some $j \in \mathbb{N}$, (2) will be satisfied for d' with $C = \sqrt[j]{8}$; i.e. for any $C > 1$, there is a quasi-metric compatible with $\mathcal{U}$ such that (2) holds with this C.

The topology induced by a balanced quasi-metric is completely regular [2], while a quiet quasi-uniformity induces a regular topology (Doitchinov, cited in [4]). Considering the similarity of the two notions, it is somewhat surprising that, as shown by an example of Fletcher, Hejcman and Hunsaker [4], a quiet quasi-uniformity is not necessarily completely regular.

The aim of this note is to give a similar example, which, in addition, is quasi-metrizable. Observe that the two notions are now even closer to each other: compare (1) and (2), where, as we have seen, one can take $C = 1 + \varepsilon$.

2. The example

Let $\mathbb{Q}$ denote the set of the rationals, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $Q_i =]i, i+1[\cap \mathbb{Q}$ $(i \in \mathbb{N}_0)$, $\mathcal{E}$ the Euclidean topology on $\mathbb{Q}$. For a convergent sequence s in $\mathbb{Q}$, denote its $\mathcal{E}$-limit by $\lambda(s)$. Let an injection $\nu : \mathbb{Q} \to \mathbb{N}$ be fixed. Take a maximal almost disjoint collection A_0 of strictly decreasing sequences in Q_0 that $\mathcal{E}$-converge to some point in Q_0. For a sequence $s = < x_j >$ in Q_0, and for each $i \in \mathbb{N}_0$, let $s + i = < x_j + i >$. Define $A_i = \{s + i : s \in A_0\}$ $(i \in \mathbb{N})$, $A = \bigcup_1^\infty A_i$, $Q = \bigcup_0^\infty Q_i$, $X = \{\omega\} \cup Q \cup A$. For $i \in \mathbb{N}$ and $s = < x_j > \in A_i$, let $s^* = < 2i - x_j >$. Now $A_i^* = \{s^* : s \in A_i\}$ is a maximal almost disjoint collection of increasing sequences in Q_{i-1} that $\mathcal{E}$-converge to some point in Q_{i-1}, while A_{i-1} is a similar collection of decreasing sequences, and $A_{i-1} \cup A_i^*$ is clearly almost disjoint, too. Define a function d on $X \times X$ as follows:

$$d(x, y) = \begin{cases} 1/i & \text{if } x = \omega,\ y \in A_i \cup Q_{i-1},\ i \in \mathbb{N}, \\ y - \lambda(x) & \text{if } y \in x \in A,\ \nu(y) > \nu(\lambda(x)), \\ \lambda(x^*) - y & \text{if } x \in A,\ y \in x^*,\ \nu(y) > \nu(\lambda(x^*)), \\ 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

Claim 1. *d is a non-Archimedian quasi-metric.*

Proof. In the second line of the definiton, y is in a decreasing sequence tending to $\lambda(x)$, hence the value of d is positive in this case, and similarly in the third line. So we have only to check that

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

This is evident if the right hand side is 1 or $x = y$ or $y = z$; otherwise $x = \omega$, $y \in A_i$ for some $i \in \mathbb{N}$, and $z \in Q_{i-1} \cup Q_i$, thus the right hand side is $\geq 1/i$, and the left hand side is $1/i$ or $1/(i+1)$. $\diamondsuit$

Claim 2. *The topology $\mathcal{T}(d)$ is regular.*

Proof. All the open balls round points different form ω are closed (because if x_1, x_2 are distinct points in A then $x_1 \cup x_1^*$ and $x_2 \cup x_2^*$ are almost disjoint), while

$$\left\{ \{\omega\} \cup \bigcup_{i=j}^{\infty} (A_i \cup Q_i) : j \in \mathbb{N} \right\}$$

is a neighbourhood base of ω consisting of closed sets. $\diamondsuit$

Claim 3. *Any $\mathcal{T}(d)$-neighbourhood of the $\mathcal{T}(d)$-closure of an interval in Q_i contains an interval in Q_{i-1}.*

Proof. Let $\emptyset \neq H =]a, b[\cap Q_i$, $a^* = 2i - a$, $b^* = 2i - b$, $H^* =]b^*, a^*[\cap Q_{i-1}$, F the $\mathcal{T}(d)$-closure of H, G a $\mathcal{T}(d)$-neighbourhood of F. Now $s \in F$ whenever $s \in A_i$ with $\lambda(s) \in H$, thus each $t \in A_i^*$ with $\lambda(t) \in H^*$ is almost contained by G.

Assume indirectly that G does not contain a subinterval of H^*. Then we can pick a strictly increasing sequence in $H^* \setminus G$ that $\mathcal{E}$-converges to some point of H^*, contradicting the maximality of the almost disjoint collection A_i^*. $\diamondsuit$

Claim 4. *$\mathcal{T}(d)$ is not completely regular.*

Proof. If f is a continuous real function on X, and $f(\omega) > 0$ then, according to Claim 3, there is a $q \in Q_0$ with $f(q) > 0$; thus ω and Q_0 cannot be separated by a continuous function. $\diamondsuit$

Claim 5. *The topology $\mathcal{T}(d^{-1})$ is discrete.*

Proof. For $y \in Q_i$, choose $\varepsilon > 0$ such that $z \in]y - \varepsilon, y + \varepsilon[\cap (Q_i \setminus \{y\}) \Rightarrow \nu(z) > \nu(y)$, and assume also that $\varepsilon < 1/(i+1)$. Then the d^{-1}-ball of radius ε round y is equal to $\{y\}$ (see the condition on $\nu(y)$ in the definiton of d). The points outside Q are evidently isolated. $\diamondsuit$

Claim 6. *If* (f, g) *is a* $\mathcal{U}(d)$*-Cauchy filter pair then* $\{z\} \in \mathsf{f}$ *for some* $z \in X$*, and* g $\mathcal{T}(d)$*-converges to* z*.*

Proof. It is enough to show that f contains a singleton, because the second assertion is then clear from the definition of the Cauchy property.

Choose $F \in \mathsf{f}$ and $G \in \mathsf{g}$ such that

$$d(x, y) < 1 \qquad (x \in F,\ y \in G). \tag{3}$$

Now if F is finite then f contains a smallest element F_0, and, according to the Cauchy property, g $\mathcal{T}(d)$-converges to each element of F_0, i.e. F_0 is a one-point set, since $\mathcal{T}(d)$ is T_2. On the other hand, if F is infinite then there are different points $x_1, x_2 \in F \cap A$, because (3) implies that $|F \cap Q| \leq 1$. Thus from (3) we have

$$G \subset (\{x_1\} \cup x_1 \cup x_1^*) \cap (\{x_2\} \cup x_2 \cup x_2^*) \in \mathsf{g},$$

and this intersection is finite by the almost disjointness, i.e. there is a point $z \in \bigcap \mathsf{g}$. According to the Cauchy property, f $\mathcal{T}(d^{-1})$-converges to z, and then Claim 5 implies that $\{z\} \in \mathsf{f}$. $\diamondsuit$

Claim 7. *The quasi-uniformity* $\mathcal{U}(d)$ *is quiet.*

Proof. Let $U_j = \{(x, y) : d(x, y) \leq 1/j\}$. We are going to show that the condition in the defintion of quietness holds for $U = U_j$ and $V = U_{j+1}$. Take a filter pair (f, g) with $\{z\} \in \mathsf{f}$ and g $\mathcal{T}(d)$-converging to z (by Claim 6, all the Cauchy filter pairs are of this form). We have to show that if $U_{j+1}x \in \mathsf{g}$ and $U_{j+1}^{-1}y \in \mathsf{f}$ then xU_jy; this is a consequence of the following statement: if

$$d(x, y_n) \leq 1/(j+1) \qquad (n \in \mathrm{N}), \tag{4}$$

$$d(z, y_n) < 1/n \qquad (n \in \mathrm{N}), \tag{5}$$

$$d(z, y) \leq 1/(j+1) \tag{6}$$

then

$$d(x, y) \leq 1/j. \tag{7}$$

It is indeed enough to prove that (4), (5) and (6) imply (7): if $U_{j+1}x \in \mathsf{g}$ then points $y_n \in U_{j+1}x$ satisfying (5) can be chosen because g converges to z, and then (4) holds evidently; moreover, $U_{j+1}^{-1}y \in \mathsf{f}$ implies (6), and xU_jy is equivalent to (7).

If $z = \omega$ then (4) and (5) imply $x = \omega$, thus (7) follows from (6). If $z \in Q$ then $y = z = y_1$ by (6) and (5), thus (7) follows from (4). Finally, assume that $z \in A_i$ for some $i \in \mathbb{N}$. From (4), (5) and the almost disjointness we have $x = z$ or $x = \omega$. (6) implies (7) in the first case; on the other hand, if $x = \omega$ then, by (5), $y_1 \in Q_{i-1} \cup Q_i \cup A_i$, thus $d(x, y_1)$ is either $1/i$ or $1/(i+1)$, hence (4) implies $i \geq j$; according to (6), $y \in Q_{i-1} \cup Q_i \cup A_i$, thus $d(x,y) \leq 1/i \leq i/j$. ◇

Remarks. a) Similarly to the example in [4], our example is complete in the sense of Doitchinov [3]. (Clear from Claim 6, since, by definition, the completeness of $\mathcal{U}$ means that the second element of any Cauchy filter pair is $\mathcal{T}(\mathcal{U})$-convergent.)

b) The topology $\mathcal{T}(d)$ can be regarded as a special case of a general construction from [7], and it was very likely described long before. (*Added in proof.* See the addition in proof in [7]).

References

[1] DEÁK, J.: Extensions of quasi-uniformities for prescribed bitopologies II, *Studia Sci. Math. Hungar.*, (to appear).

[2] DOITCHINOV, D.: On completeness in quasi-metric spaces, *Topology Appl.* **30** (1988), 127 – 145.

[3] DOITCHINOV, D.: On completeness of quasi-uniform spaces, *C. R. Acad. Bulg. Sci.* **41** (1988) No 7, 5 – 9.

[4] FLETCHER, P; HEJCMAN, J. and HUNSAKER, W.: A non-completely regular quiet quasi-uniformity, *Proc. Amer. Math. Soc.*, (to appear).

[5] FLETCHER, P. and LINDGREN, W.F.: Quasi-uniform spaces, Marcel Dekker, New York, 1982.

[6] KELLEY, J.L: General topology, Van Nostrand, New York, 1955.

[7] TZANNES, V.: Spaces containing two points not separated by a continuous real-valued function, *Topology, Theory and Applications II* (Proc. Sixth Colloq., Pécs, 1989), Colloq. Math. Soc. J. Bolyai **55**, North-Holland, Amsterdam, (to appear).