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ON THE INVARIANT MEASURE FOR JACOBI-PERRON ALGORITHM

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Abstract: Since the days of Gauss it has been known that continued fraction algorithm admits an invariant measure. Its density may be written in the form $\rho(x) = \int (1 + x\xi)^{-2} d\xi$. The aim of this paper is to give an explicit expression for the density of 2-dimensional Jacobi-Perron algorithm. The result is given as $\rho(x, y) = \int\int (1 + f(\xi, \eta)x + g(\xi, \eta)y)^{-3} d\kappa(\xi, \eta)$ where the functions f and g are given by a limiting processs and κ is a singular measure.

0. Introduction

The Jacobi-Perron algorithm was introduced by C. G. Jacobi 1868 and later generalized by O. Perron 1907. The main point was the attempt to extend Lagrange's theorem to algebraic numbers of higher degree, namely to characterize algebraic numbers by the periodicity of the algorithm. In spite of several efforts the problem still waits for its solution (see L. Bernstein 1971, Bouhamza 1984).

The ergodic theory for the Jacobi-Perron algorithm was developped along the lines already known for continued fractions (Schweiger 1973). For continued fractions the density of the (up to a constant factor) unique invariant measure which is equivalent to Lebesgue measure has been known implicitly since the days of Gauss:

$$\rho(x) = \frac{1}{1+x}. \tag{0.1}$$

Let

$$Tx = \frac{1}{x} - [\frac{1}{x}] \tag{0.2}$$

be the map associated with continued fractions then for any measurable set

$$\int_{T^{-1}E} \rho(x)dx = \int_E \rho(x)dx.$$

The associated transfer operator is given by

$$(\mathcal{A}\psi)(x) = \sum_{k=1}^{\infty} \psi\left(\frac{1}{k+x}\right)\frac{1}{(k+x)^2}. \tag{0.3}$$

The invariant density ρ then is characterized by the property $\mathcal{A}\rho = \rho$.

For the Jacobi-Perron algorithm it can be shown that there exists a finite invariant measure μ which is equivalent to Lebesgue measure but no explicit expression comparable with (0.1) has been known. The paper aims to give an explicit expression which is however more complicated. Our approach will explain why this is to be expected. In order to illustrate the basic ideas more clearly we restrict the discussion to the case $n = 2$ but the arguments are valid in the general case.

1. A heuristic approach

We first consider continued fractions (see Khintchine 1963). Given a sequence $(k_1, k_2, \dots, k_n)$ of digits we define p_n and q_n by

$$\begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} = \prod_{j=1}^{n} \begin{pmatrix} 0 & 1 \\ 1 & k_j \end{pmatrix}.$$

Then

$$(\mathcal{A}^n 1)(x) = \sum_{(n)} (q_n + q_{n-1}x)^{-2} = \\ = \sum_{(n)} q_n^{-2} \left(1 + \frac{q_{n-1}}{q_n} x\right)^{-2}.$$

Here the sum runs over all admissible sequences $(k_1, k_2, \dots, k_n)$. Next we define a seqence of measures (ν_n), $n \geq 1$, by

$$\nu_n(\psi) := \sum_{(n)} q_n^{-2} \psi \left(\frac{q_{n-1}}{q_n}\right).$$

If $\nu = \lim_{n\to\infty} \nu_n$ exists, then

$$\rho(x) := \int_0^1 \frac{d\nu(z)}{(1+zx)^2}$$

should be the density of an invariant measure.
It is well known that

$$\frac{q_{n-1}}{q_n} = [k_n, k_{n-1}, \dots, k_1].$$

We introduce $K_j := k_{n+1-j}$, $1 \leq j \leq n$, and we define sequences (P_n), (Q_n), $n = 1, 2, \dots$ by

$$\begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix} = \prod_{j=1}^{n} \begin{pmatrix} 0 & 1 \\ 1 & K_j \end{pmatrix}.$$

Then $q_{n-1} = P_n$ and $q_n = Q_n$. We define

$$f_n := \frac{P_n}{Q_n} = \frac{q_{n-1}}{q_n}$$

and for $\xi = [K_1, K_2, \ldots]$

$$f_n(\xi) := f_n$$

in an obvious manner. Then $\lim_{n\to\infty} f_n(\xi) = \xi$. The measure ν_n may be written as

$$\nu_n(\psi) = \sum_{(n)} Q_n^{-2}\psi(f_n).$$

Since $f_n \in \mathcal{B}(K_1, \ldots, K_n)$, the cylinder determined by the digits $K_1, \ldots, K_n$, it looks like a Riemann sum.
In fact one can prove

$$d\nu = \frac{d\lambda}{\log\ 2}$$

where λ denotes Lebesgue measure.
Now we consider the Jacobi-Perron algorithm for $n = 2$. The associated map is given by

$$T(x, y) = \left(\frac{y}{x} - a_1, \frac{1}{x} - b_1\right),$$

$$a_1 = a_1(x, y) := \left[\frac{y}{x}\right], \qquad b_1 = b_1(x, y) := \left[\frac{1}{x}\right].$$

If $a_j(x, y) := a_1(T^{j-1}(x, y))$, $b_j(x, y) := b_1(T^{j-1}(x, y))$ then this sequence of digits is subject to the following conditions:

$$(1.1) \qquad 1 \le b_j, \qquad 0 \le a_j \le b_j;$$

$$(1.2) \qquad \text{if} \quad a_j = b_j, \quad \text{then} \quad 1 \le a_{j+1}.$$

Similarly one defines p_n, r_n, q_n by

$$(1.3) \qquad \begin{pmatrix} p_{n-2} & p_{n-1} & p_n \\ r_{n-2} & r_{n-1} & r_n \\ q_{n-2} & q_{n-1} & q_n \end{pmatrix} := \prod_{j=1}^{n} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & a_j \\ 0 & 1 & b_j \end{pmatrix}.$$

Then

$$\lim_{n\to\infty}\left(\frac{p_n}{q_n},\frac{r_n}{q_n}\right)=(x,y).$$

We put

$$\mathcal{B}_1 := \{(x,y) : 0 \le x < y \le 1\},$$
$$\mathcal{B}_2 := \{(x,y) : 0 \le y \le x \le 1\}.$$

Then the transfer operator $\mathcal{A}$ is given by

$$(1.4)\qquad \begin{aligned}(\mathcal{A}\psi)(x,y) &= \sum_{b=1}^{\infty}\sum_{a=0}^{b}\psi\left(\frac{1}{b+y},\frac{a+x}{b+y}\right)\frac{1}{(b+y)^3} \quad \text{on } \mathcal{B}_1\\ (\mathcal{A}\psi)(x,y) &= \sum_{b=1}^{\infty}\sum_{a=0}^{b-1}\psi\left(\frac{1}{b+y},\frac{a+x}{b+y}\right)\frac{1}{(b+y)^3} \quad \text{on } \mathcal{B}_2.\end{aligned}$$

Therefore

$$(1.5)\qquad \begin{aligned}(\mathcal{A}^n 1)(x,y) &= \sum_{(n)}(q_n+q_{n-1}y+q_{n-2}x)^{-3} =\\ &= \sum_{(n)} q_n^{-3}\left(1+\frac{q_{n-1}}{q_n}y+\frac{q_{n-2}}{q_n}x\right)^{-3}.\end{aligned}$$

Here the sum runs over all admissible sequences (a_j, b_j), $1 \le j \le n$ and depends on (x,y). Again $\rho(x,y)$ is the density of an invariant measure if and only if ρ satisfies Kuzmin's equation $\mathcal{A}\rho = \rho$. We define a sequence of measure (ν_n), $n \ge 1$ by

$$(1.6)\qquad \nu_n(\psi) := \sum_{(n)} q_n^{-3}\psi\left(\frac{q_{n-1}}{q_n},\frac{q_{n-2}}{q_n}\right).$$

If again $\nu := \lim_{n\to\infty}\nu_n$ exists then

$$\rho(x,y) = \int\int\frac{d\nu(z,w)}{(1+zy+wx)^3}$$

(where the domain of integration depends on (x, y)) should be the density of an invariant measure. To understand the following construction we note that

$$(1.7)\qquad \begin{aligned} \frac{q_{n-1}}{q_n} &= \left(b_n + a_n\frac{q_{n-2}}{q_{n-1}} + \frac{q_{n-3}}{q_{n-1}}\right)^{-1}, \\ \frac{q_{n-2}}{q_n} &= \frac{q_{n-2}}{q_{n-1}}\left(b_n + a_n\frac{q_{n-2}}{q_{n-1}} + \frac{q_{n-3}}{q_{n-1}}\right)^{-1}. \end{aligned}$$

2. The basic construction

We now introduce a modified Jacobi-Perron algorithm. Actually it is a change in notation only.

$$S(\xi,\eta) := \left(\frac{\eta}{\xi} - B_1 + A_1,\ \frac{1}{\xi} - B_1\right),$$

$$B_1 = B_1(\xi,\eta) := \left[\frac{1}{\xi}\right],\quad A_1 = A_1(\xi,\eta) = B_1 - \left[\frac{\eta}{\xi}\right],$$

$$B_j(\xi,\eta) := B_1(S^{j-1}(\xi,\eta)),\quad A_j(\xi,\eta) := A_1(S^{j-1}(\xi,\eta)).$$

Then this sequence of digits is subject to the following conditions:

$$(2.1)\qquad 1 \le B_j, \qquad 0 \le A_j \le B_j;$$

$$(2.2)\qquad \text{if } A_j = 0, \qquad \text{then } A_{j+1} < B_{j+1}.$$

The most important fact is the following observation:
Define $(A_j, B_j) := (a_{n+1-j}, b_{n+1-j})$, $1 \le j \le n$, then the sequence (a_j, b_j), $1 \le j \le n$, is admissible for T if and only if the sequence (A_j, B_j), $1 \le j \le n$, is admissible for S, in short: S is a dual or backward algorithm for T (see Schweiger 1979, Ito 1986). Similarly to (1.3) we define P_n, R_n, Q_n by

$$(2.3)\qquad \begin{pmatrix} P_{n-2} & P_{n-1} & P_n \\ R_{n-2} & R_{n-1} & R_n \\ Q_{n-2} & Q_{n-1} & Q_n \end{pmatrix} = \prod_{j=1}^{n} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & B_j - A_j \\ 0 & 1 & B_j \end{pmatrix}.$$

Again the property

$$\lim_{n\to\infty} \left(\frac{P_n}{Q_n}, \frac{R_n}{Q_n}\right) = (\xi, \eta)$$

holds. Note that

$$S(\xi', \eta') = (\xi, \eta)$$

is equivalent to

$$(\xi', \eta') = \left(\frac{1}{B+\eta}, \frac{B_1 - A_1 + \xi}{B+\eta}\right)$$

for appropriate A_1 and B_1.

Next we define sequences α_{jn}, β_{jn}, $0 \le j \le 2$, $n \ge 1$ by the matrix relation

$$\text{(2.4)} \qquad \begin{pmatrix} \alpha_{1,n-1} & \beta_{1n} & \alpha_{1n} \\ \alpha_{2,n-1} & \beta_{2n} & \alpha_{2n} \\ \alpha_{0,n-1} & \beta_{0n} & \alpha_{0n} \end{pmatrix} = \prod_{j=1}^{n} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & A_j & B_j \end{pmatrix}.$$

If $(a_{n+1-j}, b_{n+1-j}) = (A_n, B_n)$, $1 \le j \le n$, then

$$\text{(2.5)} \qquad q_{n-2} = \alpha_{1n}, \quad q_{n-1} = \alpha_{2n}, \quad q_n = \alpha_{0n}.$$

Note that ν_n may be rewritten as

$$\text{(2.6)} \qquad \nu_n(\psi) = \sum_{(n)} \alpha_{0n}^{-3} \psi \left(\frac{\alpha_{2n}}{\alpha_{0n}}, \frac{\alpha_{1n}}{\alpha_{0n}}\right).$$

Here the sum runs over all admissible sequences (A_j, B_j), $1 \le j \le n$, <u>or</u> over all admissiable sequences (A_j, B_j), $1 \le j \le n$, with the initial condition $A_1 < B_1$.

If $\widehat{\alpha}_{jn}, \widehat{\beta}_{jn}$, $0 \le j \le 2$, $n \ge 1$, are given by

$$\begin{pmatrix} \widehat{\alpha}_{1,n-1} & \widehat{\beta}_{1n} & \widehat{\alpha}_{1n} \\ \widehat{\alpha}_{2,n-1} & \widehat{\beta}_{2n} & \widehat{\alpha}_{2n} \\ \widehat{\alpha}_{0,n-1} & \widehat{\beta}_{0n} & \widehat{\alpha}_{0n} \end{pmatrix} = \prod_{j=2}^{n} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & A_j & B_j \end{pmatrix},$$

then

$$(2.7)\qquad \begin{aligned} \frac{\alpha_{2n}}{\alpha_{0n}} &= \left(B_1 + A_1 \frac{\widehat{\alpha}_{2n}}{\widehat{\alpha}_{0n}} + \frac{\widehat{\alpha}_{1n}}{\widehat{\alpha}_{0n}} \right)^{-1}, \\ \frac{\alpha_{1n}}{\alpha_{0n}} &= \frac{\widehat{\alpha}_{2n}}{\widehat{\alpha}_{0n}} \left(B_1 + A_1 \frac{\widehat{\alpha}_{2n}}{\widehat{\alpha}_{0n}} + \frac{\widehat{\alpha}_{1n}}{\widehat{\alpha}_{0n}} \right)^{-1}. \end{aligned}$$

The relation (2.7) reflects (1.7).
For any finite admissible sequence (A_j, B_j), $1 \le j \le n$, we define

$$f_n := \frac{\alpha_{2n}}{\alpha_{0n}}, \quad g_n := \frac{\alpha_{1n}}{\alpha_{0n}}.$$

If $(\xi, \eta) = ((A_j, B_j),\ j \ge 1)$ is the expansion of (ξ, η) into an infinite Jacobi-Perron algorithm we define

$$f_n(\xi, \eta) = f_n, \quad g_n(\xi, \eta) = g_n$$

in the obvious way. Then we can prove the following:

Lemma 1. *The following limit exists:*

$$\lim_{n \to \infty} (f_n(\xi, \eta), g_n(\xi, \eta)) =: (f(\xi, \eta), g(\xi, \eta)).$$

Proof. We follow an idea of R.Fischer 1972. Put $\pi_n := (f_n, g_n)$. Then

$$\pi_{n+1} = \kappa_n \pi_n + \lambda_n \pi_{n-1} + \mu_n \pi_{n-2}$$

where the weights are given by

$$\begin{aligned} \kappa_n &= \frac{B_{n+1}\alpha_{0n}}{B_{n+1}\alpha_{0n} + A_n \alpha_{0,n-1} + \alpha_{0,n-2}} \\ \lambda_n &= \frac{A_n \alpha_{0,n-1}}{B_{n+1}\alpha_{0n} + A_n \alpha_{0,n-1} + \alpha_{0,n-2}} \\ \mu_n &= \frac{\alpha_{0,n-2}}{B_{n+1}\alpha_{0n} + A_n \alpha_{0,n-1} + \alpha_{0,n-2}}. \end{aligned}$$

Therefore π_{n+1} lies in the triangle spanned by π_{n-2}, π_{n-1} and π_n. Since $\alpha_{0n} = B_n \alpha_{0,n-1} + \beta_{0,n-1}$ clearly $\kappa_n \ge \frac{1}{3}$. Similarly

$$\pi_{n+2} = \kappa'_n \pi_n + \lambda'_n \pi_{n-1} + \mu'_n \pi_{n-2}.$$

Here $\kappa'_n = \kappa_n \kappa_{n+1} \geq \frac{1}{9}$.
Put $\delta(n) := \max(||\pi_n - \pi_{n-1}||, ||\pi_n - \pi_{n-2}||, ||\pi_{n-1} - \pi_{n-2}||)$.
The function $\delta(n)$ is decreasing. We see

$$||\pi_n - \pi_{n+1}|| \leq (1 - \kappa_n)\delta(n),$$

$$||\pi_{n+1} - \pi_{n+2}|| \leq (1 - \kappa_{n+1})\delta(n+1),$$

$$||\pi_n - \pi_{n+2}|| \leq (1 - \kappa'_n)\delta(n).$$

Hence $\delta(n+2) \leq \frac{8}{9}\delta(n)$.

Lemma 2. *If*

$$(\xi', \eta') = \left(\frac{1}{B+\eta}, \frac{B - A + \xi}{B + \eta}\right)$$

then

$$\begin{aligned} f(\xi', \eta') &= (B + Af(\xi, \eta) + g(\xi, \eta))^{-1}, \\ g(\xi', \eta') &= f(\xi, \eta)(B + Af(\xi, \eta) + g(\xi, \eta))^{-1}. \end{aligned} \tag{2.8}$$

Proof. This follows immediately from (2.7).

Remark. *The functions f and g are not continuous and not injective. In the case of continued fractions the corresponding function reduces to the identity $f(\xi) = \xi$.*

With the help of the functions f and g we now define the sequence of measures (κ_n) on $\mathcal{B}^*$ by

$$\nu_n(\psi) =: \int_{\mathcal{B}^*} \psi(f_n(\xi, \eta), g_n(\xi, \eta)) d\kappa_n(\xi, \eta)$$

and the measure κ on $\mathcal{B}^*$ by

$$\nu(\psi) =: \int_{\mathcal{B}^*} \psi(f(\xi, \eta), g(\xi, \eta)) d\kappa(\xi, \eta)$$

(if the limit measure exists).

Lemma 3. *If $\nu = \lim_{n\to\infty} \nu_n$ exists then*

$$d\kappa(\xi',\eta') = \frac{d\kappa(\xi,\eta)}{(B + Af(\xi,\eta) + g(\xi,\eta))^3}. \tag{2.9}$$

Proof. This again follows from

$$\alpha_{0n} = B_1\widehat{\alpha}_{0n} + A_1\widehat{\alpha}_{2n} + \widehat{\alpha}_{1n}.$$

Remark. *If the map $(\xi,\eta) \mapsto (f(\xi,\eta), g(\xi,\eta))$ is absolutely continuous then* it is easily seen that

$$d\kappa(\xi,\eta) := \frac{\partial(f,g)}{\partial(\xi,\eta)} d\xi \, d\eta$$

has the transformation property (2.9).

3. The invariant measure

The heuristic considerations in section 1 now suggest:

Theorem 1. *The density of the invariant measure μ is given as follows:*

$$\rho(x,y) = \iint\limits_{\mathcal{B}^*} \frac{d\kappa(\xi,\eta)}{(1 + f(\xi,\eta)y + g(\xi,\eta)x)^3} \quad \text{for} \quad (x,y) \in \mathcal{B}_1,$$
$$\rho(x,y) = \iint\limits_{\mathcal{B}_1^*} \frac{d\kappa(\xi,\eta)}{(1 + f(\xi,\eta)y + g(\xi,\eta)x)^3} \quad \text{for} \quad (x,y) \in \mathcal{B}_2.$$

Here

$$\mathcal{B}_1^* = \{(\xi,\eta) : 0 \le \xi < \eta \le 1\},$$
$$\mathcal{B}_2^* = \{(\xi,\eta) : 0 \le \eta \le \xi \le 1\},$$
$$\mathcal{B}^* = \mathcal{B}_1^* \cup \mathcal{B}_2^*.$$

Proof. Remember that

$$\mathcal{B}^*(a,b) = \{(\xi,\eta) : A_1(\xi,\eta) = a,\ B_1(\xi,\eta) = b\}.$$

Then

$$\begin{aligned}
\mathcal{B}_1^* &= \bigcup_{b=1}^{\infty} \bigcup_{a=0}^{b-1} \mathcal{B}^*(a,b), \\
\mathcal{B}_2^* &= \bigcup_{b=1}^{\infty} \mathcal{B}^*(b,b), \\
S\mathcal{B}^*(a,b) &= \mathcal{B}^* \ \text{ if } \ a \geq 1, \\
S\mathcal{B}^*(0,b) &= \mathcal{B}_1^*.
\end{aligned}$$

The following identity is basic:

$$(1 + f(\xi,\eta)\frac{x+a}{y+b} + g(\xi,\eta)\frac{1}{y+b})^3\ (y+b)^3 =$$

$$= \left(1 + \frac{y}{b + af(\xi,\eta) + g(\xi,\eta)} + \frac{xf(\xi,\eta)}{b + af(\xi,\eta) + g(\xi,\eta)}\right)^3 \cdot$$

$$\cdot (b + af(\xi,\eta) + g(\xi,\eta))^3.$$

Therefore we obtain (by use of relation (2.9))

$$\iint_{\mathcal{B}^*} \frac{d\kappa(\xi,\eta)}{(1 + f(\xi,\eta)\frac{x+a}{y+b} + g(\xi,\eta)\frac{1}{y+b})^3\ (y+b)^3} =$$

$$= \iint_{\mathcal{B}^*(a,b)} \frac{d\kappa(\xi',\eta')}{(1 + f(\xi',\eta')y + g(\xi',\eta')x)^3} \qquad \text{if} \quad a \geq 1$$

and

$$\iint_{\mathcal{B}_1^*} \frac{d\kappa(\xi,\eta)}{(1 + f(\xi,\eta)\frac{x}{y+b} + g(\xi,\eta)\frac{1}{y+b})^3 (y+b)^3} =$$

$$= \iint_{\mathcal{B}^*(0,b)} \frac{d\kappa(\xi',\eta')}{(1 + f(\xi',\eta')y + g(\xi',\eta')x)^3}.$$

Since

$$\mathcal{B}^* = \bigcup_{b=1}^{\infty} \bigcup_{a=1}^{b} \mathcal{B}^*(a,b) \cup \bigcup_{b=1}^{\infty} \mathcal{B}^*(0,b),$$

$$\mathcal{B}_1^* = \bigcup_{b=1}^{\infty} \bigcup_{a=1}^{b-1} \mathcal{B}^*(a,b) \cup \bigcup_{b=1}^{\infty} \mathcal{B}^*(0,b),$$

a comparison with (1.4) shows that ρ satisfies Kuzmin's equation $A\rho = = \rho$.

Remark. *The map*

$$\varepsilon(x,y,\xi,\eta) = \left(\frac{y}{x} - a, \frac{1}{x} - b, \frac{1}{b+\eta}, \frac{b-a+\xi}{b+\xi}\right)$$

can be seen as the natural extension of T (see Ito 1986; Bosma-Jager--Wiedijk 1983). *The measure*

$$\frac{dx\ dy\ d\kappa(\xi,\eta)}{(1+f(\xi,\eta)y+g(\xi,\eta)x)^3}$$

is invariant with respect to ε.

4. κ is singular

As before we denote by μ the invariant absolutely continuous probability measure for the map T. We introduce a new set function τ as follows:

$$\tau(\mathcal{B}((a_1,b_1),\dots,(a_n,b_n))) := \mu(\mathcal{B}((b_n-a_n,b_n),\dots,(b_1-a_1,b_1))).$$

Then it is checked easily that τ is in fact an invariant measure for T. Furthermore τ also is an ergodic measure. Therefore $\tau = \mu$ or τ and μ are mutually singular. We will prove that $\tau = \mu$ is impossible.

Theorem 2. *The measure* τ *is singular with respect to* μ.

Proof. Suppose the contrary, namely $\tau = \mu$. This means

$$\mu(\mathcal{B}((a_1,b_1),\ldots,(a_n,b_n)) = \mu(\mathcal{B}((b_n - a_n,b_n),\ldots,(b_1 - a_1,b_1)))$$

for all admissible sequences.
It is well known that there is a constant $D \geq 1$ such that

$$D^{-1}q_n^{-3} \leq \mu(\mathcal{B}((a_1,b_1),\ldots,(a_n,b_n))) \leq Dq_n^{-3}$$

and hence

$$D^{-1}Q_n^{-3} \leq \mu(\mathcal{B}((b_n - a_n,b_n),\ldots,(b_1 - a_1,b_1))) \leq DQ_n^{-3}.$$

If in fact $\mu = \tau$ we obtain

$$D^{-2} \leq \left(\frac{Q_n}{q_n}\right)^3 \leq D^2. \tag{4.1}$$

But this is impossible. It is sufficient to take $(a_j,b_j) = (0,1)$, $1 \leq j \leq n$. Then

$$q_n \sim \alpha^n \text{ and } Q_n \sim \beta^n$$

where α is the greatest root of $x^3 - x^2 - 1 = 0$ and β is the greatest root of $x^3 - x^2 - x - 1 = 0$. Then $1 < \alpha < \beta$ and (4.1) cannot be true.

Remark. *However the corresponding entropies coincide:*

$$h(\mu,T) = h(\tau,T).$$

Corollary.

$$\lim_{n\to\infty} \frac{Q_n}{\alpha_{0n}} = 0 \qquad \textit{for almost all} \quad (\xi,\eta) \in \mathcal{B}^*.$$

Proof. Since τ is singular with respect to μ the martingale convergence theorem shows that

$$\lim_{n\to\infty} \frac{\tau(B((a_1,b_1),\ldots,(a_n,b_n))}{\mu(B((a_1,b_1),\ldots,(a_n,b_n))} = \lim_{n\to\infty} \left(\frac{q_n}{Q_n}\right)^3 = 0$$

for (Lebesgue) almost all $(x, y) \in \mathcal{B}$. Therefore by symmetry we obtain

$$\lim_{n\to\infty} \left(\frac{Q_n}{q_n}\right)^3 = 0$$

for (Lebesgue) almost all $(\xi, \eta) \in \mathcal{B}^*$.
Finally note that $\alpha_{0n} = q_n$.

Remark. In the case of continued fractions it is well known that $Q_n = = q_n$. Therefore $\mu(\mathcal{B}(k_1, \ldots, k_n))) = \mu(\mathcal{B}(k_n, \ldots, k_1)))$ *for all admissible sequences.*

Lemma. *The limits* $\nu = \lim_{n\to\infty} \nu_n$ *resp.* $\kappa = \lim_{n\to\infty} \kappa_n$ *exist.*

Proof. It is sufficient to show that

$$\lim_{n\to\infty} \nu_n(\psi) =: \nu(\psi)$$

exists for any function ψ which is Lipschitz continuous on $\mathcal{B}^*$.
The map $\Phi(\xi, \eta) := (f(\xi, \eta),\ g(\xi, \eta))$ does not satisfy a Lipschitz condition generally, but it satisfies an appropriate condition on cylinders. The proof of lemma 1 shows that if

$$(\xi, \eta),\ (\xi^*, \eta^*) \in \mathcal{B}^*((A_1, B_1), \ldots, (A_n, B_n)))$$

then

$$||\Phi(\xi, \eta) - \Phi(\xi^*, \eta^*)|| << \delta(n)$$

hence

$$|\psi(\Phi(\xi, \eta)) - \psi(\Phi(\xi^*, \eta^*))|| << \delta(n).$$

An examination of the proof of Kuzmin's theorem for Jacobi-Perron algorithm shows that the Lipschitz condition is used on cylinders only (Schweiger - Waterman 1973). Therefore

$$\sum_{(n)} \psi \left(\Phi \left(\frac{p_n + p_{n-1}y + p_{n-2}x}{q_n + q_{n-1}y + q_{n-2}x}, \frac{r_n + r_{n-1}y + r_{n-2}x}{q_n + q_{n-1}y + q_{n-2}y} \right) \right) \cdot$$

$$\cdot \frac{1}{(q_n + q_{n-1}y + q_{n-2}x)^3} \to \rho(x, y)c(\psi) \quad \text{as} \quad n \to \infty.$$

Note that

$$\Phi\left(\frac{p_n}{q_n},\frac{r_n}{q_n}\right) = \left(\frac{\alpha_{2n}}{\alpha_{0n}},\frac{\alpha_{1n}}{\alpha_{0n}}\right).$$

Since the summation $\sum_{(n)}$ depends on $(x,y)\in\mathcal{B}_1$ resp. $(x,y)\in\mathcal{B}_2$ we take $\lim_{(x,y)\to(0,0)}$ with $(x,y)\in\mathcal{B}_1$ resp. $\lim_{(x,y)\to(0,0)}$ with $(x,y)\in\mathcal{B}_2$.

If we write L_1 for the first limit and L_2 for the second limit then we obtain as $n\to\infty$

$$\sum_{(n)}\psi\left(\frac{\alpha_{2n}}{\alpha_{0n}},\frac{\alpha_{1n}}{\alpha_{0n}}\right)\frac{1}{\alpha_{0n}^3} \to L_1\, c(\psi)$$

if the sum runs over all admissible sequences $((A_1,B_1),\dots,(A_n,B_n))$ and

$$\sum_{(n)}\psi\left(\frac{\alpha_{2n}}{\alpha_{0n}},\frac{\alpha_{1n}}{\alpha_{0n}}\right)\frac{1}{\alpha_{0n}^3} \to L_2\, c(\psi)$$

if the sum runs over all admissible sequences $((A_1,B_1),\dots,(A_n,B_n))$ with the additional condition $A_1<B_1$.

Theorem 3. *The limit measure κ is singular with respect to Lebesgue measure on $\mathcal{B}^*$.*

Proof. Since

$$\lambda(\mathcal{B}^*((A_1,B_1),\dots,(A_n,B_n))) \sim Q_n^{-3},$$

this follows at once from Theorem 2 (applied to the measure τ transposed to S) or its Corollary.

Remark. Since ε maps the set $\mathcal{B}((a_1,b_1),\dots,(a_n,b_n))\times\mathcal{B}^*$ onto $\mathcal{B}\times\mathcal{B}^*((b_n-a_n,b_n),\dots,(b_1-a_1,b_1)))$ (if $a_n<b_n$), well known theorems from ergodic theory (see Friedman 1970) show that ε *does not admit a finite invariant measure equivalent to Lebesgue measure on $\mathcal{B}\times\mathcal{B}^*$.*

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