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SIMULTANEOUS EXTENSIONS OF PROXIMITIES, SEMI-UNIFORMITIES, CONTIGUITIES AND MEROTOPIES I*

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Abstract: Given compatible proximities (in the sense of Čech) on some subspaces of a closure space, we are looking for a common compatible extension of these proximities. In Part II, proximities will be replaced by semi-uniformities, contiguities or merotopies. In Parts III and IV, we shall consider similar extension problems in proximity, semi-uniform and contiguity spaces.

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We are going to investigate problems of the following type: Let X be a set, σ a topological structure (e.g. a closure) on $X, \{X_i : i \in I\}$ a system of subsets of X; assume that a richer structure (e.g. a proximity) Σ_i is given on each X_i; we aim at finding a common *extension* of these structures, i.e. a structure Σ compatible with σ such that $\Sigma|X_i = \Sigma_i$ $(i \in I)$, where $\Sigma|X_i$ denotes the restriction of Σ to X_i. Two natural necessary conditions for the existence of such an extension: (i) Σ_i has to be compatible with $\sigma|X_i$, and (ii) $\Sigma_i|X_i \cap X_j = \Sigma_j|X_i \cap X_j$ $(i, j \in I)$ [assuming, of course, that for arbitrary structures σ and Σ on X, and for $B \subset A \subset X$, (i) $\Sigma|A$ is compatible with $\sigma|A$ whenever Σ is compatible with σ, and (ii) $\Sigma|B = (\Sigma|A)|B$; these conditions will be evidently satisfied in each particular case we are going to consider].

See [13] for a survey of the classical extension problem when $|I| = 1$.

§0 contains all the necessary definitions and notation (including those needed only in Parts II to IV). §1 deals with the case when σ is a closure and Σ a proximity.

In Part II, σ will be again a closure, and Σ a semi-uniformity, a contiguity or a merotopy.

In Part III, σ will be a proximity, and Σ a contiguity or a merotopy.

The following cases will be investigated in Part IV: a) σ is a proximity, Σ a semi-uniformity, b) σ is a semi-uniformity or a contiguity, Σ a merotopy.

Each of the above mentioned questions will be considered in three variants: a) without separation axioms; b) for Riesz-type structures; c) for Lodato-type structures.

These problems clearly have category theoretical aspects, which will not be investigated here. It would be interesting to find out the category theoretical reasons for the similarity of some results, and for the dissimilarity of others, cf. [13] Problem 72.

0. Preliminaries

All the unproved statements in this section are either well-known or trivial (usually both).

0.1 Closures. A *closure* [2] on X is a function $c : \exp X \to \exp X$ such that, for $A, B \subset X$,

C1. $c(\emptyset) = \emptyset$,
C2. $A \subset c(A)$,
C3. $A \subset B$ implies $c(A) \subset c(B)$,
C4. $c(A \cup B) \subset c(A) \cup c(B)$.

If, in additon, $c(c(A)) = c(A)$ for every $A \subset X$ then c is a *topology*.

The closure c is said to be *symmetric* [27] (semi-uniformizable in [2]) if $y \in c(\{x\})$ implies $x \in c(\{y\})$ for $x, y \in X$; it is *separated* [7] (semi-separated in [2], D_1 in [27]) if $c(\{x\}) = \{x\}$ for $x \in X$, and *weakly separated* [8] if $x \notin c(A)$ implies $c(\{x\}) \cap c(A) = \emptyset$. A symmetric closure is weakly separated iff $x \in c(A)$ implies $c(\{x\}) \subset c(A)$; this condition is Axiom H_2 in [27]. Separated implies weakly separated, which in turn implies symmetric. A topology is separated iff it is T_1, and weakly separated iff it is symmetric iff it is S_1 in the sense of [6] (better known as R_0, but we shall use the term *S_1-topology*).

If c is a closure on X , and $x \in X$ then a *c-neighbourhood* [2] of x is a set $V \subset X$ such that $x \notin c(X \setminus V)$; the c-neighbourhoods of x constitute the *c-neighbourhood filter* of x; a *c-neighbourhood (sub)base* of x is a (sub)base for the c-neighbourhood filter of x. (Occasionally, when there is no danger of confusion, the letter c will be dropped from these names; the same convention applies to other notions depending on some structure.) For $A \subset X$, $\mathrm{int}_c A$ denotes the set of all $x \in X$ such that A is a neighbourhood of x. $\mathrm{int}\, A = X \setminus c(X \setminus A)$.

If c and c' are closures on X then c is said to be *coarser* than c' (c' *finer* than c) if $c'(A) \subset c(A)$ for $A \subset X$.

For $X_0 \subset X$, the *restriction* to X_0 of the closure c, denoted by $c|X_0$, is defined by $c_0(A) = c(A) \cap X_0$ $(A \subset X_0)$, where $c_0 = c|X_0$; c_0 is a closure on X_0, symmetric, (weakly) separated or topological if c is so. If c' is finer than c then $c'|X_0$ is finer than $c|X_0$.

Denoting the c-neighbourhood filter of $x \in X$ by $\mathsf{v}(x)$, we say that $\mathsf{s}_0(x) = \mathsf{v}(x)|X_0$ is the *trace filter* (on X_0) of the point x, where, for $\mathsf{s} \subset \exp X$,

$$\mathsf{s}|X_0 = \{S \cap X_0 : S \in \mathsf{s}\},$$

called the *trace* (on X_0) of s. For $x \in X_0$, $\mathsf{s}_0(X)$ coincides with the c_0-neighbourhood filter of x, while $\mathsf{s}_0(x) = \exp X_0$ (the *zero filter* on

X_0) whenever $x \notin c(X_0)$. This means that, in general, only the trace filters of the points in $c(X_0) \setminus X_0$ will be of interest.

0.2 Proximities. A *proximity* [2] (called basic proximity or Čech proximity when the shorter term is reserved for proximities in the sense of Efremovich) on X is a relation $\delta \subset \exp X \times \exp X$ such that, for $A, B, C, A', B' \subset X$

P1. $A\delta B$ implies $B\delta A$,
P2. $A\delta X$ implies $A \neq \emptyset$,
P3. $A \cap B \neq \emptyset$ implies $A\delta B$,
P4. $A\delta B$, $A \subset A'$, $B \subset B'$ imply $A'\delta B'$,
P5. $(A \cup B)\delta C$ implies that either $A\delta C$ or $B\delta C$.

We write $\bar{\delta}$ for non-δ. Parantheses will often be omitted, e.g.: $A \cup B\delta C$.

The relation β is a *base* for the relation δ (this is in fact a sub-base-like notion) provided that

$$A\bar{\delta}B \text{ iff there are } n, m \in \mathrm{N} \text{ and sets } A_i, B_j \subset X$$
$$(1 \leq i \leq n,\ 1 \leq j \leq m) \text{ such that } A_i\bar{\beta}B_j$$
$$\text{for each } i \text{ and } j,\ A = \bigcup_1^n A_i,\ B = \bigcup_1^m B_j.$$

(N denotes the set of the positive integers.) Clearly, $\delta \subset \beta$. If β is a base for δ, and β satisfies Axioms P1 to P4 then δ is a proximity; any proximity is a base for itself.

A proximity δ *induces* a symmetric closure $c = c(\delta)$ defined by

$$x \in c(A) \text{ iff } \{x\}\delta A.$$

The proximity δ is said to be *Riesz* [26] (SP$''$ in [7], weakly Lodato in [8]) if, with $c = c(\delta)$

PRi. $A\bar{\delta}B$ implies $c(A) \cap c(B) = \emptyset$,

and *Lodato* [25] (P_s-relation in [23]) if

PLo. $A\bar{\delta}B$ implies $c(A)\bar{\delta}c(B)$.

PLo implies PRi. δ is Riesz or Lodato iff there is a base β for δ such that $A\bar{\beta}B$ implies $c(A) \cap c(B) = \emptyset$, respectively $c(A)\bar{\beta}c(B)$ [$c(A)\bar{\delta}c(B)$]. If δ is Riesz (Lodato) then $c(\delta)$ is weakly separated (it is an S_1-topology).

For proximities δ and δ' on X, δ is said to be *coarser* than δ' (δ' *finer* than δ) if $\delta \supset \delta'$. If β is a base for δ, β' for δ', and $\beta \supset \beta'$ then $\delta \supset \delta'$; in particular, if β is a base for δ, δ' is a proximity, and $\bar{\beta} \subset \bar{\delta}'$ then δ is coarser than δ'. The finest proximity on X is called *discrete* ($A\delta B$ iff $A \cap B \neq \emptyset$); the coarsest one is called *indiscrete* ($A\delta B$ iff $A \neq \emptyset \neq B$). A finer proximity induces a finer closure.

If $X_0 \subset X$, the *restriction* $\beta_0 = \beta|X_0$ of the relation β is defined for $A, B \subset X_0$ by $A\beta_0 B$ iff $A\beta B$. If β is a base for δ then $\beta|X_0$ is a base for $\delta|X_0$. The restriction of a (Riesz/Lodato) proximity is again a (Riesz/Lodato) proximity. For a proximity δ, $\delta|X_0$ induces $c(\delta)|X_0$. The restriction of a finer proximity is finer.

[If $\beta_0 = \beta|X_0$, we write $\bar{\beta}_0$ for non-β_0 in X_0; this notation cannot be misunderstood if our attention is restricted to relations β satisfying axioms P1 to P4 (or just P2 and $X\beta X$ if $X \neq \emptyset$), because then β, as well as $\bar{\beta}$, determines the fundamental set: it is $\bigcup \mathrm{dom}\beta = \bigcup \mathrm{dom}\bar{\beta}$].

A filter $\mathfrak{s}$ on X is said to be *δ-compressed* [6,7] (or: $\mathfrak{s}$ is a compressed filter in the proximity space (X, δ)) if $A, B \subset X$, $A, B \in \sec \mathfrak{s}$ imply $A\delta B$, where

$$\sec \mathfrak{s} = \sec_X \mathfrak{s} = \{A \subset X : \ A \cap S \neq \emptyset \ (S \in \mathfrak{s})\}.$$

The zero filter is compressed. A proximity δ is Riesz iff each $c(\delta)$-neighbourhood filter is δ-compressed. If $\mathfrak{s}$ is δ-compressed then $\mathfrak{s}|X_0$ is $\delta|X_0$-compressed.

0.3 Semi-uniformities.

A *semi-uniformity* [2] on X is a filter $\mathcal{U}$ on $X \times X$ such that

U1. each $U \in \mathcal{U}$ is an *entourage*, i.e. $\Delta \subset U$,
U2. $U^{-1} \in \mathcal{U}$ for $U \in \mathcal{U}$,

where $\Delta = \Delta_X$ is the diagonal of X, and U^{-1} is the inverse of U:

$$\Delta_X = \{(x,x) : x \in X\}, \quad U^{-1} = \{(x,y) : yUx\},$$

and xUy means $(x,y) \in U$. For $x \in X$ and $A \subset X$ we write

$$U[A] = \{y : \exists x \in A, xUy\}, \quad Ux = U[\{x\}].$$

A *(sub)base* for a semi-uniformity is to be understood as a filter (sub)base on $X \times X$. The symmetric entourages contained by the semi-uniformity $\mathcal{U}$ form a base for $\mathcal{U}$. Any non-empty collection $\mathcal{S}$ of entourages is a subbase for a semi-uniformity, provided that for each $U \in \mathcal{U}$, U^{-1} contains some $V \in \mathcal{S}$; in particular, any non-empty collection of symmetric entourages is a subbase for some semi-uniformity.

A semi-uniformity $\mathcal{U}$ *induces* a proximity $\delta = \delta(\mathcal{U})$ defined by

$$(1) \qquad A\delta B \text{ iff } (A \times B) \cap U \neq \emptyset \quad (U \in \mathcal{U});$$

equivalently:

$$(2) \qquad A\bar{\delta}B \text{ iff } U[A] \cap B = \emptyset \text{ for some } U \in \mathcal{U}.$$

Hence $\mathcal{U}$ *induces* a closure $c(\mathcal{U}) = c(\delta(\mathcal{U}))$. $\{Ux : U \in \mathcal{U}\}$ is the $c(\mathcal{U})$-neighbourhood filter of $x \in X$. In (1) and (2), $\mathcal{U}$ can be replaced by any base for $\mathcal{U}$. If $\mathcal{S}$ is a (sub)base for $\mathcal{U}$ then $\{Ux : U \in \mathcal{S}\}$ is a (sub)base for $\mathsf{v}(x)$ in $c(\mathcal{U})$.

The semi-uniformity $\mathcal{U}$ is said to be *Riesz* if

URi. $U \in \mathcal{U}$ implies $\Delta \subset \operatorname{int}_{c \times c} U$,

where the $(c \times c)$-neighbourhood filter of $(x, y) \in X \times X$ is generated by the filter base

$$\{G \times H : G \in \mathsf{v}(x), \quad H \in \mathsf{v}(y)\},$$

and $c = c(\mathcal{U})$. $\mathcal{U}$ is said to be *Lodato* if

ULo. $U \in \mathcal{U}$ implies $\operatorname{int}_{c \times c} U \in \mathcal{U}$.

$\mathcal{U}$ is Riesz (Lodato) iff URi (ULo) holds with $\mathcal{U}$ replaced by a subbase; $\mathcal{U}$ is Lodato iff it has a (sub)base consisting of open entourages. (A set A is *c-open* if $A = \operatorname{int} A$; an *open entourage* is meant to be $(c(\mathcal{U}) \times c(\mathcal{U}))$-open.) ULo implies URi. If $\mathcal{U}$ is Riesz (Lodato) then so is $\delta(\mathcal{U})$. URi and ULo fit naturally between the corresponding axioms for proximities and merotopies, so they are probably known; nevertheless, we are unable to cite a source.

For two semi-uniformities $\mathcal{U}$ and $\mathcal{U}'$ on X, $\mathcal{U}$ is said to be *coarser* than $\mathcal{U}'$ ($\mathcal{U}'$ *finer* than $\mathcal{U}$) if $\mathcal{U} \subset \mathcal{U}'$; in this case $\delta(\mathcal{U})$ is coarser than $\delta(\mathcal{U}')$.

For $X_0 \subset X$, the *restriction* $\mathcal{U}|X_0$ of the semi-uniformity $\mathcal{U}$ to X_0 is defined by

$$\mathcal{U}|X_0 = \{U|X_0 : U \in \mathcal{U}\}, \quad U|X_0 = U \cap (X_0 \times X_0).$$

$\mathcal{U}|X_0$ is a semi-uniformity on X_0 satisfying $\delta(\mathcal{U}|X_0) = \delta(\mathcal{U})|X_0$. If $\mathcal{U}$ is Riesz or Lodato then so is $\mathcal{U}|X_0$. The restriction of a finer semi-uniformity is finer.

A filter s on X is *$\mathcal{U}$-Cauchy* if $U \in \mathcal{U}$ implies $S \times S \subset U$ for some $S \in \mathsf{s}$. ($\mathcal{U}$ can be replaced by a subbase in this definition.) If s is $\mathcal{U}$-Cauchy then it is $\delta(\mathcal{U})$-compressed, and $\mathsf{s}|X_0$ is $\mathcal{U}|X_0$-Cauchy for $X_0 \subset X$. $\mathcal{U}$ is Riesz iff every $c(\mathcal{U})$-neighbourhood filter is Cauchy.

0.4 Merotopies. A *merotopy* [21] (quasi-uniformity in [19], Čech nearness in [24]) on X is a non-empty collection M of covers of X such that

M1. if $\mathsf{c} \in \mathsf{M}$ and c refines d then $\mathsf{d} \in \mathsf{M}$,
M2. if $\mathsf{c}, \mathsf{d} \in \mathsf{M}$ then $\mathsf{c}(\cap)\mathsf{d} \in \mathsf{M}$,

where

$$\mathsf{c}(\cap)\mathsf{d} = \{C \cap D : C \in \mathsf{c},\ D \in \mathsf{d}\}.$$

($\{\emptyset\}$ is a cover of $X = \emptyset$; $\emptyset$ is not a cover of it. c *refines* d, or c is a *refinement* of d, if for any $C \in \mathsf{c}$ there is a $D \in \mathsf{d}$ with $C \subset D$.) M2 can be replaced by

M2′. any two elements of M have a common refinement in M.

A subset B of a merotopy M is a *base* for M if every element of M has a refinement in B; B satisfies Axiom M2′. Conversely, any non-empty collection B of covers that satisfies M2′ is a base for exactly one merotopy M; a cover c belongs to M iff it has a refinement in B.

For a finite non-empty family F of covers, we define $(\cap)\mathsf{F}$ as follows:

$$A \in (\cap)\mathsf{F} \text{ iff } \exists A(\mathsf{c}) \in \mathsf{c} \ (\mathsf{c} \in \mathsf{F}), \quad A = \cap\{A(\mathsf{c}) : \mathsf{c} \in \mathsf{F}\}.$$

(If $\mathsf{F} = \{\mathsf{c}, \mathsf{d}\}$ and $\mathsf{c} \neq \mathsf{d}$ then $(\cap)\mathsf{F} = \mathsf{c}(\cap)\mathsf{d}$.) A subset S of a merotopy M is a *subbase* for M if

$$\{(\cap)\mathsf{F} : \emptyset \neq \mathsf{F} \subset \mathsf{S},\ \mathsf{F} \text{ is finite}\}$$

is a base for M. Any non-empty collection of covers of X is a subbase for exactly one merotopy on X.

A merotopy M *induces* a semi-uniformity $\mathcal{U}(\mathsf{M})$, for which a base $\mathcal{B}$ (the one consisting of all the symmetric elements of $\mathcal{U}(\mathsf{M})$) is defined by

$$\mathcal{B} = \{U(\mathsf{c}) : \mathsf{c} \in \mathsf{M}\}, \quad U(\mathsf{c}) = \bigcup\{C \times C : C \in \mathsf{c}\}.$$

(Taking c from a (sub)base only, we obtain a (sub)base for $\mathcal{U}(\mathsf{M})$.) Hence M induces a proximity $\delta(\mathsf{M}) = \delta(\mathcal{U}(\mathsf{M}))$ and a closure $c(\mathsf{M}) = c(\delta(\mathsf{M}))$. For $\delta = \delta(\mathsf{M})$,

$$A\delta B \text{ iff } \mathrm{St}(A, \mathsf{c}) \cap B \neq \emptyset \quad (\mathsf{c} \in \mathsf{M}), \tag{1}$$

where

$$\mathrm{St}(A, \mathsf{c}) = \bigcup\{C \in \mathsf{c} : A \cap C \neq \emptyset\}.$$

$\{\mathrm{St}(x, \mathsf{c}) : \mathsf{c} \in \mathsf{M}\}$ is the $c(\mathsf{M})$ neighbourhood filter of x, where $\mathrm{St}(x, \mathsf{c}) = \mathrm{St}(\{x\}, \mathsf{c})$. M can be replaced by a base in (1). If S is a (sub)base for M then $\{\mathrm{St}(x, \mathsf{c}) : \mathsf{c} \in \mathsf{S}\}$ is a (sub)base for $\mathsf{v}(x)$ in $c(\mathsf{M})$.

A merotopy M on X is said to be *Riesz* (Riesz nearness in [3]) if

MRi. for each $\mathsf{c} \in \mathsf{M}$, $\mathrm{int}\, \mathsf{c}$ is a cover of X,

where

$$\mathrm{int}\, \mathsf{c} = \mathrm{int}_c\, \mathsf{c} = \{\mathrm{int}_c\, C : C \in \mathsf{c}\},$$

and $c = c(\mathsf{M})$. M is said to be *Lodato* (nearness in [16], Lodato nearness in [24]) if

MLo. $\mathsf{c} \in \mathsf{M}$ implies $\mathrm{int}\, \mathsf{c} \in \mathsf{M}$.

MLo implies MRi. M is Riesz (Lodato) iff MRi (MLo) holds for some subbase for M; M is Lodato iff it has a subbase consisting of $c(\mathsf{M})$-open covers. If M is Riesz (Lodato) then so is $\mathcal{U}(\mathsf{M})$.

For two merotopies M and M' on X, M is said to be *coarser* than M' (M' *finer* than M) if $\mathsf{M} \subset \mathsf{M}'$. If S is a subbase for M and $\mathsf{S} \subset \mathsf{M}'$ then M is coarser than M'. $\{\{X\}\}$ is a base for the *indiscrete* (coarsest) merotopy on X; the *discrete* (finest) merotopy on X consists of all the covers of X. A finer merotopy induces a finer semi-uniformity.

For $X_0 \subset X$, the *restriction* $\mathsf{M}|X_0$ of the merotopy M to X_0 is defined by

$$(2) \qquad \mathsf{M}|X_0 = \{\mathsf{c}|X_0 : \mathsf{c} \in \mathsf{M}\}.$$

$\mathsf{M}|X_0$ is a merotopy on X_0 satisfying $\mathcal{U}(\mathsf{M}|X_0) = \mathcal{U}(\mathsf{M})|X_0$. If M is replaced by a (sub)base then (2) yields a (sub)base for $\mathsf{M}|X_0$. If M is Riesz or Lodato then so is $\mathsf{M}|X_0$. The restriction of a finer merotopy is finer.

A filter s on X is M-*Cauchy* [19] if $\mathsf{s} \cap \mathsf{c} \neq \emptyset$ for $\mathsf{c} \in \mathsf{M}$ (equivalently: for $c \in \mathsf{S}$, where S is a subbase for M). M-Cauchy filters are $\mathcal{U}(M)$-Cauchy as well. If s is M-Cauchy then $\mathsf{s}|X_0$ is $\mathsf{M}|X_0$-Cauchy. M is Riesz iff every $c(\mathsf{M})$-neighbourhood filter is M-Cauchy.

0.5 Contiguities. A *contiguity* (essentially [20,17]) on X is a non-empty collection Γ of finite covers of X such that

Co1. if $\mathsf{c} \in \Gamma$, c refines d, and d is finite then $\mathsf{d} \in \Gamma$,
Co2. if $\mathsf{c}, \mathsf{d} \in \Gamma$ then $\mathsf{c}(\cap)\mathsf{d} \in \Gamma$.

Base and *subbase* for a contiguity, *Riesz* and *Lodato* contiguities, *finer* and *coarser* contiguities, the *restriction* $\Gamma|X_0$ of a contiguity, and Γ-*Cauchy* filters are defined in the same way as for merotopies. ("Contiguity" means a Lodato contiguity in [16].) The proximity $\delta = \delta(\Gamma)$ *induced* by Γ is defined by 0.4 (1) (with Γ substituted for M); $c(\Gamma) = c(\delta(\Gamma))$ is the closure *induced* by Γ. (It is superfluous to define $\mathcal{U}(\Gamma)$ in the same way as $\mathcal{U}(\mathsf{M})$, because $\mathcal{U}(\Gamma)$ is then uniquely determined by $\delta(\Gamma)$.) The analogues of all the statements for merotopies listed in 0.4 are valid for contiguities, too. In addition, if S is a subbase for Γ then

$$A\beta B \text{ iff } \mathrm{St}(A, \mathsf{c}) \cap B \neq \emptyset \quad (c \in \mathsf{S})$$

defines a base β for $\delta(\Gamma)$.

For a merotopy M, the contiguity $\Gamma(\mathsf{M})$ *induced* by M consists of the finite elements of M. If M is Riesz or Lodato then so is $\Gamma(\mathsf{M})$. $\delta(\Gamma(\mathsf{M})) = \delta(\mathsf{M})$ and $\Gamma(\mathsf{M})|X_0 = \Gamma(\mathsf{M}|X_0)$. A finer merotopy induces a finer contiguity. Any M-Cauchy filter is $\Gamma(\mathsf{M})$-Cauchy.

Contiguities as well as semi-uniformities are structures lying between merotopies and proximities. Neither of the structures $\Gamma(\mathsf{M})$ and $\mathcal{U}(\mathsf{M})$ determines the other, and they together do not determine M.

0.6 Conventions. A *family of proximities* in the closure space (X, c) is a system $\{\delta_i : i \in I\}$, where I is a (possibly empty) set of indices, such that δ_i is a proximity on some $X_i \subset X$, $X \neq \emptyset$, $X_i \neq \emptyset$ $(i \in I)$, and the two conditions mentioned in the introduction are fullfilled, i.e.

$$(1) \qquad c(\delta_i) = c|X_i \qquad (i \in I),$$

$$(2) \qquad \delta_i|X_i \cap X_j = \delta_j|X_i \cap X_j \qquad (i, j \in I).$$

Where these conditions have to be referred to, we shall say that the family of proximities is (the proximities δ_i are) (1) *compatible* and (2) *accordant.* When speaking about a *family of proximities in a closure space*, it will be understood that the closure space is denoted by (X, c), and I, δ_i and X_i are used as above; moreover, $c_i = c|X_i$, $\mathrm{int} = \mathrm{int}_c$, $\mathrm{int}_i = \mathrm{int}_{c_i}$, $\mathrm{Int} = \mathrm{int}_{c \times c}$, $\mathrm{Int}_i = \mathrm{int}_{c_i \times c_i}$, $X_{ij} = X_i \cap X_j$, $\mathsf{v}(x)$ is the c-neighbourhood filter of $x \in X$, $\mathsf{s}_i(x)$ the c-trace filter of $x \in X$ on X_i. The expression *"the trace filters are compressed"* means that $\mathsf{s}_i(x)$ is δ_i-compressed for each $x \in X$ and each $i \in I$. An *extension* of $\{\delta_i : i \in I\}$ (or of the proximities δ_i) is a proximity δ on X such that $c = c(\delta)$ and $\delta_i = \delta|X_i$ $(i \in I)$. In case the proximities δ_i have an extension, we shall also say that they can be extended.

Analogous terminology, notations and conventions will be used for other kinds of structures, with $\mathcal{U}$ and $\mathcal{U}_i$ standing for semi-uniformities, Γ and Γ_i for contiguities, M and M_i for merotopies; "compressed" will be replaced by "Cauchy". If the structure given on X is not a closure then c denotes the closure induced by it, and the notations derived from c (int, $\mathsf{s}_i(x)$, etc.) will be used as above.

1. Extending a family of proximities in a closure space

A. WITHOUT SEPARATION AXIOMS

1.1 If a family of proximities can be extended in a closure space then the closure clearly has to be symmetric. We are going to show that this condition is sufficient, too. In fact, we construct the finest and the coarsest extension.

Definition. Given a family of proximities in a closure space, define $\delta^1 \subset \exp X \times \exp X$ as follows. $A\,\delta^1 B$ iff one of the following conditions holds:

$$A \cap c(B) \neq \emptyset, \tag{1}$$

$$c(A) \cap B \neq \emptyset, \tag{2}$$

$$A \cap X_i \delta_i B \cap X_i \text{ for some} \tag{3}$$

In case confusion seems to be possible, we write $\delta^1(c, \delta_i)$, or, more precisely, $\delta^1(c, \{\delta_i : i \in I\})$; in particular, $\delta^1(c) = \delta^1(c, \emptyset)$. ◇

Theorem. *A family of proximities in a symmetric closure space always has extensions; δ^1 is the finest one.*

Proof. It is easy to see that δ^1 is a proximity on X.

1° *$\delta^1|X_i$ is coarser than δ_i.* If $A\delta_i B$ then (3) holds, and therefore $A\delta^1 B$.

2° *$\delta^1|X_i$ is finer than δ_i.* Assume $A(\delta^1|X_i)B$; this means that $A\delta^1 B$ and $A, B \subset X_i$. If (1) holds then $A \cap c_i(B) \neq \emptyset$, so there is a point $x \in A \cap c_i(B)$; hence $\{x\}\delta_i B$ by the compatibility of δ_i, thus $A\delta_i B$. The case of (2) is analogous. Finally, if (3) holds, i.e. if $A \cap X_j \delta_j B \cap X_j$ for some j then $A \cap X_j$, $B \cap X_j \subset X_{ij}$ implies $A \cap X_j \delta_i B \cap X_j$ by the accordance, so $A\delta_i B$ again.

3° *$c(\delta^1)$ is coarser than c.* If $x \in c(B)$ then (1) is satisfied with $A = \{x\}$, so $\{x\}\delta^1 B$.

4° *$c(\delta^1)$ is finer than c.* Assume $x \notin c(B)$; we have to show that none of the conditions (1) to (3) holds with $A = \{x\}$. $\{x\} \cap c(B) = \emptyset$ is evident. For $y \in B$, $x \notin c(\{y\})$, thus we have $y \notin c(\{x\})$ from the symmetry of c, and so $c(\{x\}) \cap B = \emptyset$. Finally,

$$\{x\} \cap X_i \bar{\delta}_i B \cap X_i \quad (i \in I),$$

because the left hand side is empty if $x \notin X_i$, and, for $x \in X_i$, $x \notin c(B)$ implies $x \notin c(B \cap X_i) \cap X_i = c_i(B \cap X_i)$, thus $\{x\}\bar{\delta}_i B \cap X_i$.

5° *δ^1 is the finest extension.* Let δ be another extension; we have to show that $\delta^1 \subset \delta$. Assume $A\delta^1 B$. If (1) holds then $x \in c(B)$ for some $x \in A$, thus $\{x\}\delta B$ and $A\delta B$; similarly, (2) implies $A\delta B$. Finally, from (3) we have $A \cap X_i \delta B \cap X_i$ (since $\delta|X_i = \delta_i$), hence $A\delta B$ again. ◇

1.2 Our next aim is to construct the coarsest extension; its definition will be a little bit more complicated than that of the finest one.

Definition. For a family of proximities in a closure space, let β be a base for $\delta^0 \subset \exp X \times \exp X$, where $A\bar{\beta}B$ iff one of the following conditions holds:

$$(1) \qquad |A| \leq 1 \text{ and } A \cap c(B) = \emptyset,$$

$$(2) \qquad |B| \leq 1 \text{ and } c(A) \cap B = \emptyset,$$

$$(3) \qquad A\bar{\delta}_i B \text{ for some } i \in I.$$

The notations $\delta^0(c, \delta_i)$, etc. will be used as in Definition 1.1 (and similar conventions will apply to all subsequent definitions). ◇

Theorem. *A family of proximities in a symmetric closure space always has a coarsest extension, namely* δ^0.

Proof. β clearly satisfies Axioms P1 to P4, so δ^0 is a proximity on X.

1° $\delta^0|X_i$ *is finer than* δ_i. If $A\bar{\delta}_i B$ then (3) holds, so $A\bar{\beta}B$ and $A\bar{\delta}^0 B$.

2° $\delta^0|X_i$ *is coarser than* δ_i. $\beta|X_i$ is a base for $\delta^0|X_i$, so it is enough to show that $\bar{\beta}|X_i \subset \bar{\delta}_i$. Assume that $A\bar{\beta}B$ and $A, B \subset X_i$. If (1) holds and $A \neq \emptyset$ then $A = \{x\}$ for some $x \in X_i$; now $x \notin c(B)$, so $x \notin c_i(B)$, implying $\{x\}\bar{\delta}_i B$, i.e. $A\bar{\delta}_i B$. The case of (2) is analogous. Finally, if $A\bar{\delta}_j B$ for some $j \in I$ then $A, B \subset X_{ij}$, so $A\bar{\delta}_i B$ follows from the accordance.

3° $c(\delta^0)$ *is finer than* c. If $x \notin c(B)$ then (1) holds with $A = \{x\}$, thus $\{x\}\bar{\beta}B$ and $\{x\}\bar{\delta}^0 B$.

4° $c(\delta^0)$ *is coarser than* c. Assume $\{x\}\bar{\delta}^0 B$. Then B can be written as a finite union $\bigcup_n B_n$ such that $\{x\}\bar{\beta}B_n$ for each n; it is now enough to show that

$$(4) \qquad x \notin c(B_n),$$

because then $x \notin c(B)$ by Axiom C4. If (1) holds (with $A = \{x\}$ and $B = B_n$) then (4) is evident. If (2) holds and $B_n \neq \emptyset$ then $B_n = \{y\}$, and the symmetry of c implies $x \notin c(\{y\})$, which is the same as (4). Finally, if $\{x\}\bar{\delta}_i B_n$ for some i then $x \notin c_i(B_n)$, so $x \in X_i$ and $B_n \subset X_i$ imply (4) again.

5° δ^0 *is the coarsest extension.* Let δ be another extension; it is enough to show that $\bar{\beta} \subset \bar{\delta}$. Assume $A\bar{\beta}B$. If (1) holds and $A \neq \emptyset$ then $A = \{x\}$ for some $x \in X$, and $x \notin c(B)$, implying $A\bar{\delta}B$, which follows in the same way from (2), too. Finally, if (3) holds then $A\bar{\delta}B$ again, because $\delta_i = \delta|X_i$. ◇

Part 5° of the above proof uses only one half of the assumption that δ is an extension: δ^0 is the coarsest one among those proximities δ that induce a closure finer than c, and for which $\delta|X_i$ is finer than δ_i $(i \in I)$. Similarly, δ^1 is the finest one among those proximities δ that induce a closure coarser than c, and for which $\delta|X_i$ is coarser than δ_i $(i \in I)$, see 5° in the proof of Theorem 1.1. These observations are of some interest when compared with the results of §1C.

1.3 Recall that the proximities on a fixed set form a complete lattice with respect to the relation finer/coarser, and the infimum and the supremum of the proximities $\delta[i]$ on X $(i \in I \neq \emptyset)$ can be described as follows: $\inf_i \delta[i] = \bigcup_i \delta[i]$, while $\bigcup_i \delta[i]$ is a base for $\sup_i \delta[i]$, (see e.g. [2] 38 A.1 and 38 A.5, where the infimum is called supremum, and vice versa). Infima and suprema of proximities commute with the restriction to a subset (evident) as well as with taking the induced closure ([2] 38 B.3); constructions of infima and suprema of closures are not needed here, see them e.g. in [2] 31 A.2 and 31 ex. 2.

For $i \in I$ fixed, let us write $\delta^0[i]$ for $\delta^0(c, \{\delta_i\})$, and denote by $\delta^{00}[i]$ the coarsest proximity δ on X (not necessarily compatible with c) for which $\delta|X_i = \delta_i$; this means that $A\bar{\delta}^{00}[i]B$ iff either $A\bar{\delta}_iB$ or $A = \emptyset$ or $B = \emptyset$. Now we have, for $I \neq \emptyset$,

$$(1) \qquad \delta^0 = \sup_i \delta_0[i] = \sup\{\delta^0(c), \sup_i \delta^{00}[i]\}.$$

This could be checked looking at the constructions, but in fact it is enough to know for the proof of (1) that proximities figuring in it do exist: Denote by δ' the proximity in the middle of (1), and by δ'' the one on the right hand side of it. $\delta^0 \subset \delta^0[i] \subset \delta^{00}[i]$ is evident, and so is $\delta^0[i] \subset \delta^0(c)$, therefore $\delta^0 \subset \delta' \subset \delta''$. Moreover, $c(\delta'')$ is finer than c, and $\delta''|X_i$ is finer than δ_i, because $\delta^{00}[i]|X_i = \delta_i$; hence $\delta'' \subset \delta^0$ by the remark at the end of 1.2 (and the construction of δ^0 is not really needed in that remark either: $\delta \cup \delta^0$ is an extension, so, δ^0 being the coarsest extension, we have $\delta \subset \delta \cup \delta^0 \subset \delta^0$).

Similarly, if $\delta^1[i] = \delta^1(c, \{\delta_i\})$, and $\delta^{11}[i]$ denotes the finest proximity δ on X (not necessarily compatible with c) for which $\delta|X_i = \delta_i$ ($A\delta^{11}[i]B$ iff $A \cap X_i \delta_i B \cap X_i$ or $A \cap B \neq \emptyset$) then, for $I \neq \emptyset$,

(2) $$\delta^1 = \inf_i \delta^1[i] = \inf \{\delta^1(c), \inf_i \delta^{11}[i]\}.$$

B. RIESZ PROXIMITIES IN A CLOSURE SPACE

1.4 If a family of proximities in a closure space has a Riesz extension then each proximity is Riesz, the closure is weakly separated, and the trace filters are compressed (because the neighbourhood filters have to be compressed with respect to the extension). We are going to show that these conditions are sufficient, too; there are again a finest and a coarsest extension.

Definition. For a family of Riesz proximities in a weakly separated closure space, let $\delta^1_R \subset \exp X \times \exp X$ be defined as follows: $A\delta^1_R B$ iff either

(1) $$c(A) \cap c(B) \neq \emptyset$$

or

(2) $$A \cap X_i \delta_i B \cap X_i \quad \text{for some } i. \quad \diamond$$

Lemma. *Given a family of Riesz proximities in a weakly separated closure space, δ^1_R is a compatible Riesz proximity on X; it is the finest one among those Riesz proximities δ that induce a closure coarser than c, and for which $\delta|X_i$ is coarser than δ_i ($i \in I$).*

Proof. δ^1_R is clearly a proximity on X.

1° *$\delta^1_R|X_i$ is coarser than δ_i.* If $A\delta_i B$ then (2) holds, implying $A\delta^1_R B$.

2° *$c(\delta^1_R)$ is coarser than c.* If $x \in c(B)$ then $c(\{x\}) \cap c(B) \neq \emptyset$, so $\{x\}\delta^1_R B$ by (1).

3° *$c(\delta^1_R)$ is finer than c.* Assume $x \notin c(B)$; we have to show that $\{x\}\bar{\delta}^1_R B$, i.e. that neither (1) nor (2) holds with $A = \{x\}$. $c(\{x\}) \cap$ $\cap c(B) = \emptyset$, since c is weakly separated. $\{x\} \cap X_i \bar{\delta}_i B \cap X_i$ follows as in 4° of the proof of Theorem 1.1.

4° *δ_R^1 is Riesz.* If $A\bar{\delta}_R^1 B$ then (1) does not hold, and we have already seen that $c = c(\delta_R^1)$.

5° *δ_R^1 is finest.* Let δ be another Riesz proximity with $\delta_i \subset \delta|X_i$ $(i \in I)$ and $c(\delta)$ coarser than c; we have to show that $\delta_R^1 \subset \delta$. Assume $A\delta_R^1 B$. If (1) holds then $c'(A) \cap c'(B) \neq \emptyset$ where $c' = c(\delta)$; now $A\delta B$, because δ is Riesz. If (2) holds then $A \cap X_i \delta B \cap X_i$, and so $A\delta B$ again. ◇

Theorem. *A family of Riesz proximities in a weakly separated closure space has a Riesz extension iff the trace filters are compressed; if so then δ_R^1 is the finest Riesz extension.*

Proof. In view of the lemma, it is enough to show that if the trace filters are compressed then $\delta_R^1|X_i$ is finer than δ_i $(i \in I)$. Assume $A\delta_R^1 B$, $A, B \subset X_i$. If (1) holds then, picking $x \in c(A) \cap c(B)$, we have $A, B \in \sec \mathsf{s}_i(x)$, hence $A\delta_i B$, because $\mathsf{s}_i(x)$ is δ_i-compressed. On the other hand, if $A \cap X_j \delta_j B \cap X_j$ for some j then $A\delta_i B$ by the accordance, just like in 2° of the proof of Theorem 1.1. ◇

If $\{\operatorname{int} X_i : i \in I\}$ covers X then it is not necessary to assume that the trace filters are compressed. Indeed, if $A, B \subset X_i$, $A, B \in \sec \mathsf{v}(x)$, $x \in \operatorname{int} X_j$ then $X_j \in \mathsf{v}(x)$, so $A \cap X_j$, $B \cap X_j \in \sec \mathsf{v}(x)$, implying $A \cap X_j \delta_j B \cap X_j$ (since δ_j is Riesz); hence $A\delta_i B$ by the accordance.

Corollary. *A family of Riesz proximities in a weakly separated closure space has a Riesz extension iff*

$$\delta_i \supset \delta_R^1(c)|X_i \qquad (i \in I). \tag{3}$$

Proof. The necessity is obvious. Conversely, if (3) holds then each $\mathsf{s}_i(x)$ is δ_i-compressed, because it is compressed with respect to the finer proximity $\delta_R^1(c)|X_i$; thus the theorem applies. ◇

1.5 Lemma. *If δ' and δ'' are proximities such that $c(\delta') = c(\delta'')$, δ' is Riesz, and δ'' is coarser than δ' then δ'' is Riesz, too.* ◇

Theorem. *Under the hypotheses of Theorem 1.4, δ^0 is the coarsest Riesz extension.*

Proof. Theorems 1.2 and 1.4, and the above lemma. ◇

1.6 Assume that the conditions of Theorem 1.4 are satisfied. Similarly to 1.3 (2),

$$\delta_R^1 = \inf_i \delta_R^1[i] = \inf \{\delta_R^1(c), \inf_i \delta^{11}[i]\}, \tag{1}$$

where $\delta_R^1[i] = \delta_R^1(c, \{\delta_i\})$. Just like the other proximities in (1), $\delta^{11}[i]$ is Riesz, since, with c_i' standing for $c(\delta^{11}[i])$, we have $c_i'(A) = A \cup c_i(A \cap \cap X_i)$. Concerning 1.3 (1), let us observe that $\delta^{00}[i]$ cannot be replaced by the "coarsest Riesz proximity δ on X for which $\delta|X_i = \delta_i$", because such a proximity may not exist: let $|X| = 3$, $|X_0| = 2$, and δ_0 be the discrete proximity on X_0.

1.7 Observe that $A\bar{\delta}^0(c)B$ iff either A is finite and $A \cap c(B) = \emptyset$ or B is finite and $c(A) \cap B = \emptyset$. The next lemma will be needed in §1C.

Lemma. *If c is and S_1-topology then $\delta_R^1(c)$ is Lodato; if c is a T_1-topology then $\delta^0(c)$ is Lodato as well.*

Proof. The first statement is evident. To prove the second one, assume that c is a T_1-topology, and $A\bar{\delta}^0(c)B$. Then, say, A is finite and $A \cap \cap c(B) = \emptyset$; hence $c(A) = A$ is finite, $c(c(B)) = c(B)$, so $c(A)\ \bar{\delta}^0(c)c(B)$. ◇

C. LODATO PROXIMITIES IN A CLOSURE SPACE

1.8 If a family of proximities in a closure space has a Lodato extension then each proximity is Lodato, the closure is an S_1-topology, and the trace filters are compressed (because a Lodato proximity is Riesz). Somewhat suprisingly, these conditions are not sufficient:

Example. Let $X = \mathrm{R}^2$, c be the Euclidean topology on X, $X_0 = \mathrm{R} \times \times \{0\}$, $X_1 = X \setminus X_0$, δ_0 the Euclidean proximity on X_0, and $\delta_1 = = \delta_R^1(c)|X_1$. Now c is an S_1-topology, δ_i is a Lodato proximity compatible with $c_i (i = 0, 1)$, for $i = 1$ by Lemma 1.7. Moreover, the trace filters are compressed, since the Euclidean proximity on X is a Lodato extension of δ_0, while $\delta_R^1(c)$ is a Lodato extension of δ_1.

Assume that the family $\{\delta_0, \delta_1\}$ has a Lodato extension δ. With $\mathrm{N}' = \{n + 2^{-n} : n \in \mathbb{N}\}$, consider $A = \mathbb{N} \times (\mathbb{R} \setminus \{0\})$ and $B = \mathrm{N}' \times (\mathbb{R} \setminus$

$\setminus\{0\}$). Now $c(A) = \mathbb{N} \times \mathbb{R}$, $c(B) = \mathbb{N}' \times \mathbb{R}$, hence $A\bar{\delta}_1 B$, and so $A\bar{\delta}B$. On the other hand, $c(A) \cap X_0 \delta_0 c(B) \cap X_0$ so that $c(A) \cap X_0 \delta c(B) \cap X_0$ and $c(A)\delta c(B)$, a contradiction. $\diamondsuit$

1.9 Definition. For a family of Lodato proximities in an S_1-space, let $\delta_L^1 \subset \exp X \times \exp X$ be defined as follows: $A\delta_L^1 B$ iff either

$$(1) \qquad c(A) \cap c(B) \neq \emptyset$$

or

$$(2) \qquad c(A) \cap X_i \delta_i c(B) \cap X_i \quad \text{for some } i. \; \diamondsuit$$

Lemma. *For a family of Lodato proximities in an S_1-space, δ_L^1 is a compatible Lodato proximity; it is the finest one among those Lodato proximities δ on X that induce a closure coarser than c, and for which $\delta|X_i$ is coarser than δ_i ($i \in I$).*

Proof. It is easy to see that δ_L^1 is a proximity on X.

1^o *$\delta_L^1|X_i$ is coarser than δ_i.* If $A\delta_i B$ then (2) holds, and so $A\delta_L^1 B$.

2^o *$c(\delta_L^1)$ is coarser than c.* Just like in the proof of Lemma 1.4.

3^o *$c(\delta_L^1)$ is finer than c.* Assume $x \notin c(B)$; we have to show that neither (1) nor (2) holds with $A = \{x\}$. $c(\{x\}) \cap c(B) = \emptyset$ because c is S_1.

$$(3) \qquad c(\{x\}) \cap X_i \bar{\delta}_i c(B) \cap X_i$$

is evident if the left hand side is empty. Otherwise, one can take $y \in$ $\in c(\{x\}) \cap X_i$; now $c(\{x\}) = c(\{y\})$ (since c is S_1), thus (3) is equivalent to

$$(4) \qquad c_i(\{y\})\bar{\delta}_i c(B) \cap X_i.$$

$x \notin c(B)$ implies $y \notin c(B)$ (again by S_1), therefore $y \notin c(B) \cap X_i =$ $= c_i(c(B) \cap X_i)$, i.e. $\{y\}\bar{\delta}_i c(B) \cap X_i$, and so (4) holds indeed (as δ_i is Lodato).

4^o *δ_L^1 is Lodato.* This is clear from $c = c(\delta_L^1)$, since (1) and (2) depend only on $c(A)$ and $c(B)$, and c is a topology.

5^o *δ_L^1 is finest.* Let δ be another Lodato proximity with $\delta_i \subset$ $\subset \delta|X_i$ ($i \in I$) and $c(\delta)$ coarser than c; we have to show that $\delta_L^1 \subset \delta$. Assume $A\delta_L^1 B$. (1) implies $c'(A) \cap c'(B) \neq \emptyset$ where $c' = c(\delta)$, thus

(δ being Lodato) we have $A\delta B$. On the other hand, if (2) holds then $c(A) \cap X_i \delta c(B) \cap X_i$, so $c(A)\delta c(B)$, implying $A\delta B$ again. $\diamondsuit$

1.10 Definition. For a family of Lodato proximities in an S_1-space, let β be a base for $\delta_L^0 \subset \exp X \times \exp X$, where $A\bar{\beta}B$ iff one of the following conditions holds:

$$(1) \qquad A \subset c(\{x\}) \text{ for some } x \notin c(B), \text{ or } A = \emptyset,$$

$$(2) \qquad B \subset c(\{x\}) \text{ for some } x \notin c(A), \text{ or } B = \emptyset,$$

$$(3) \qquad \text{there are } i, A', B' \text{ with } A'\bar{\delta}_i B', A \subset c(A'), B \subset c(B'). \ \diamondsuit$$

Lemma. *If a family of Lodato proximities is given in an S_1-space, and the trace filers are compressed then δ_L^0 is the coarsest one among those compatible Lodato proximities δ on X for which $\delta|X_i$ is finer than δ_i $(i \in I)$.*

Proof. 1^o *δ_L^0 is a proximity.* β clearly satisfies Axioms P1, P2 and P4. To prove P3, assume $A\bar{\beta}B$. If (1) or (2) holds then $A \cap B = \emptyset$ follows from S_1. If (3) holds then $c(A') \cap c(B') = \emptyset$, because the trace filters are compressed; hence $A \cap B = \emptyset$ again, i.e. β fullfills P1 to P4. Consequently, δ_L^0 is a proximity indeed.

2^o *$\delta_L^0|X_i$ is finer than δ_i.* If $A\bar{\delta}_i B$ then (3) holds with $A' = A$ and $B' = B$, so $A\bar{\beta}B$ and $A\bar{\delta}_L^0 B$.

3^o *$c(\delta_L^0)$ is finer than c.* If $x \notin c(B)$ then (1) holds with $A = \{x\}$, thus $\{x\}\bar{\delta}_L^0 B$.

4^o *$c(\delta_L^0)$ is coarser than c.* Just as in 4^o of the proof of Theorem 1.2, it is enough to show that $\{y\}\bar{\beta}B$ implies

$$(4) \qquad y \notin c(B).$$

If (1) holds (with $A = \{y\}$) then $x \notin c(B)$ and S_1 imply (4). If (2) holds and $B \neq \emptyset$ then from $x \notin c(\{y\})$ and S_1 we have $y \notin c\{x\})$, which implies (4), since $c(\{x\}) = c(B)$ by S_1. Finally, if (3) holds then $y \in c(A')$, $B \subset c(B')$ and $A'\bar{\delta}_i B'$, thus $c(A') \cap c(B') = \emptyset$ (because the trace filters are compressed), and $y \notin c(B') = c(c(B')) \supset c(B)$.

5^o *δ_L^0 is Lodato.* If $A\bar{\beta}B$ then $c(A)\bar{\beta}c(B)$ follows directly from the definition (taking into account that c is a topology). Now δ_L^0 is Lodato, since we have already seen that $c = c(\delta_L^0)$.

6^o *δ_L^0 is coarsest.* Let δ be another compatible Lodato proximity with $\delta|X_i \subset \delta_i$ $(i \in I)$; it is enough to show that $\bar{\beta} \subset \bar{\delta}$. If (1) holds then either $A = \emptyset$, in which case $A\bar{\delta}B$ is evident, or $\{x\}\bar{\delta}B$ (since δ is compatible), hence $c(\{x\})\bar{\delta}B$ (since δ is Lodato), and so $A\bar{\delta}B$. The case of (2) is analogous. Finally, if (3) holds then $A'\bar{\delta}B'$, therefore $c(A')\bar{\delta}c(B')$, and $A\bar{\delta}B$ again. ◇

It is not true that δ_L^0 is the coarsest one among those Lodato proximities δ that induce a closure finer than c, and for which $\delta|X_i$ is finer than δ_i $(i \in I)$, not even when $I = \emptyset$:

Example. Let (X, c) be the topological sum of two infinite indiscrete spaces, and c' the discrete closure on X. Now c' is finer than c, but $\delta_L^0(c') = \delta^0(c')$ is not finer than $\delta_L^0(c)$, since there are infinite sets A and B with $A\bar{\delta}_L^0(c)B$, while $A\delta_L^0(c')B$ for any pair of inifinite sets. ◇

1.11 Lemma. *A family of Lodato proximities in an S_1-space has a Lodato extension iff $\delta_L^1 \subset \delta_L^0$; if so then both δ_L^0 and δ_L^1 are Lodato extensions.*

Proof. 1° *Necessity.* If δ is a Lodato extension then $\delta_L^1 \subset \delta \subset \delta_L^0$ by Lemmas 1.9 and 1.10 (the latter can be applied since the existence of δ implies that the trace filters are compressed).

2° *Sufficiency.* If $A\bar{\delta}_iB$ for some i then $c(A)\bar{\delta}_L^0c(B)$ by 1.10 (3), so $c(A)\bar{\delta}_L^1c(B)$, implying $c(A) \cap c(B) = \emptyset$ (because δ_L^1 is a proximity by Lemma 1.9); this means that the trace filters are compressed and so Lemma 1.10 applies as well as Lemma 1.9. Consequently, δ_L^0 and δ_L^1 are compatible Lodato proximities, $\delta_L^0|X_i \subset \delta_i \subset \delta_L^1|X_i$, and from $\delta_L^1 \subset \delta_L^0$ we have also $\delta_L^1|X_i \subset \delta_L^0|X_i$. Hence both δ_L^0 and δ_L^1 are extensions. ◇

Theorem. *A family of Lodato proximities in an S_1-space has a Lodato extension iff the trace filters are compressed, and, for any $i, j \in I$,*

$$A\bar{\delta}_iB \Rightarrow c(A) \cap X_j\bar{\delta}_jc(B) \cap X_j; \tag{1}$$

if so then δ_L^0 is the coarsest and δ_L^1 is the finest Lodato extension.

Remark: Observe that (1) is a strengthening of the accordance.

Proof. 1° *Necessity.* If δ is a Lodato extension then $A\bar{\delta}_iB$ implies $A\bar{\delta}B$, hence $c(A)\bar{\delta}c(B)$ and $c(A) \cap X_j\bar{\delta}c(B) \cap X_j$, thus the right hand side of (1) holds.

2° *Sufficiency.* In consequence of Lemma 1.9, it is enough to prove that $\delta_L^1|X_i$ is finer than δ_i $(i \in I)$. Assume that $A\delta_L^1B$ and $A, B \subset X_i$.

If 1.9 (1) holds then $A\delta_i B$, because the trace filters are compressed. On the other hand, if 1.9 (2) holds, i.e. if $c(A) \cap X_j \delta_j c(B) \cap X_j$ for some j then we have $A\delta_i B$ from (1).

3° δ_L^0 and δ_L^1 are Lodato extensions by the foregoing lemma; they are coarsest, respectively finest by Lemmas 1.10 and 1.9. ◇

Corollary. *A family of proximities in an S_1-space has a Lodato extension iff $\{\delta_i, \delta_j\}$ has a Lodato extension for any $i, j \in I$.* ◇

1.12 Corollary. *A single Lodato proximity given in an S_1-space has a Lodato extension iff the trace filters are compressed.*

Proof. 1.11 (1) is always satisfied for $i = j$, because $c(S) \cap X_i = c_i(S)$ $(S \subset X_i)$, and δ_i is Lodato. ◇

1.13 Theorem. *Let a family of Lodato proximities be given in an S_1-space, assume that the trace filters are compressed, and*

(1) $$c(X_i \setminus X_j) \cap (X_j \setminus X_i) = \emptyset \quad (i, j \in I).$$

Then there exists a Lodato extension.

Proof. We have to show that 1.11 (1) holds. Assume $A\bar{\delta}_i B$; it is enough to consider the following three cases because then Axioms C4 and P5 can be applied:

a) $A, B \subset X_i \setminus X_j$;
b) $A, B \subset X_{ij}$;
c) $A \subset X_i \setminus X_j,\ B \subset X_{ij}$.

Case a). From (1) we have $c(A) \cap X_j \subset X_{ij}$ and $c(B) \cap X_j \subset X_{ij}$, so, by the accordance, it is enough to prove that $c(A) \cap X_j \bar{\delta}_i c(B) \cap X_j$, which is true, because $c(A) \cap X_j \subset c(A) \cap X_i = c_i(A)$, similarly, $c(B) \cap$ $\cap X_j \subset c_i(B)$, and δ_i is Lodato.

Case b). The accordance implies $A\bar{\delta}_j B$, so the right hand side of 1.11 (1) holds again, now because δ_j is Lodato.

Case c). As in Case a), $c(A) \cap X_j \subset c_i(A)$, so $c(A) \cap X_j \bar{\delta}_i B$ (because δ_i is Lodato). The accordance implies $c(A) \cap X_j \bar{\delta}_j B$, therefore $c(A) \cap X_j \bar{\delta}_j c_j(B)$ (because δ_j is Lodato); $c_j(B) = c(B) \cap X_j$ completes the proof. ◇

Corollary. *Let a family of Lodato proximities be given in an S_1-space. Assume that either each X_i is open and the trace filters are compressed or each X_i is closed. Then there exists a Lodato extension.*

Proof. $c(X_i \setminus X_j) \cap (X_j \setminus X_i)$ does not change if c is replaced by $c|X_i \cup X_j$; $X_i \setminus X_j$ and $X_j \setminus X_i$ are disjoint closed (or open) sets in $(X_i \cup X_j, c|X_i \cup \cup X_j)$, thus (1) holds. ◇

If the sets X_i form an open cover of X then we do not have to assume that the trace filters are compressed, see after Theorem 1.4.

1.14 Assume that a non-empty family of Lodato proximities is given in an S_1-space. Similarly to 1.3 (2) and 1.6 (1), we have

(1) $$\delta_L^1 = \inf_i{}_L \delta_L^1[i] = \inf_L\{\delta_L^1(c), \inf_i{}_L \delta^{11}[i]\},$$

where $\delta_L^1[i] = \delta_L^1(c, \{\delta_i\})$, and $\inf_L$ denotes the infimum in the realm of the Lodato proximities (recall that the Lodato proximities on X form a complete lattice, see e.g. [7] (5.1); observe that $\delta^{11}[i]$ is Lodato). The proof is the same as that of 1.3 (1) and 1.3 (2). The proximity in the middle of (1) can be written as $\inf_i \delta_L^1[i]$, because the infimum of Lodato proximities inducing the same closure is Lodato, too. However, the right hand side of (1) cannot be replaced by $\inf\{\delta_L^1(c), \inf_i \delta^{11}[i]\}$:

Example. Let X, c, X_0, δ_0, A and B be as in Example 1.8, $I = \{0\}$. Then $A\bar{\delta}_L^1(c)B$, $A\bar{\delta}^{11}[0]B$, but $A\delta_L^1 B$. ◇

If, in addition, the trace filters are compressed then

(2) $$\delta_L^0 = \sup_i \delta_L^0[i].$$

(The supremum of Lodato proximities is always Lodato, see e.g. [7] (5.1).) An analogue of the right hand side of 1.3 (1) cannot be added to (2), because, in general, there is no coarsest Lodato proximity δ on X for which $\delta|X_i = \delta_i$ (see the example at the end of 1.6).

1.15 If the conditions of Theorem 1.11 are satisfied then we have the following five extensions:

(1) $$\delta^0 \supset \delta_L^0 \supset \delta_L^1 \supset \delta_R^1 \supset \delta^1.$$

If $I = \emptyset$ then $\delta^1_R = \delta^1_L$, and, assuming also that c is T_1, $\delta^0 = \delta^0_L$ (Lemma 1.7). If c is T_1, and each X_i is closed then $\delta^0 = \delta^0_L$ (look at the definitions); similarly, if each X_i is open then $\delta^1_L = \delta^1_R$. This observation yields an alternative proof of Corollary 1.13 (only in T_1-spaces if the subsets are closed, but then we can get rid of T_1 using a stock argument: switch over to the T_0-reflexion of (X, c), take an extension there, and carry it back to (X, c)).

All the proximities in (1) can be, however, different if $|I| = 1$, even when c is T_1:

Examples. a) In Example 1.14, $A\delta^1_L B$, but $A\bar{\delta}^1_R B$.

b) Let X, c, X_1, δ_1, A and B be as in Example 1.8, $I = \{1\}$. Then $c(A) \setminus X_1 \delta^0 c(B) \setminus X_1$, but $c(A) \setminus X_1 \bar{\delta}^0_L c(B) \setminus X_1$. $\diamond$

1.16 Concerning extensions of a single *Efremovich proximity*, see [22], [15] 3.25, [9] §4, [1], [10] §2, [12] 2.2., [14]. We know only the following about simultaneously extending Efremovich proximities:

a) If $\{\delta_1, \delta_2\}$ is a family of Efremovich proximities in a topological space, $X = X_1 \cup X_2$, either X_1 and X_2 are both open and the trace filters are round, or X_1 and X_2 are both closed then $\{\delta_1, \delta_2\}$ has an Efremovich extension; this follows from [13] Remark 1.13 c). (A filter $\mathfrak{s}$ in the proximity space (X, c) is *round* [22] if for any $S \in \mathfrak{s}$ there is an $S_0 \in \mathfrak{s}$ with $S_0 \bar{\delta} X \setminus S$.)

b) The above statement is false for three proximities, even if the subspaces are open-closed. (Essentially [13] Example 1.13b).)

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