Mathematica Pannonica
1/2 (1990), 45 – 49

ON RINGS SATISFYING CERTAIN POLYNOMIAL IDENTITIES*

I. Gy. **Maurer**

Mathematical Institute of the Hungarian Academy of Sciences, H-1053 Budapest, Reáltanoda u. 13-15, Hungary.

J. **Szigeti**

Mathematical Institute, University of Miskolc, H-3515 Miskolc-Egyetemváros, Hungary.

Received August 1989

AMS Subject Classification: 16 A 38, 16 A 70

Keywords: Jacobson's identity of rings, (m,n)-Boolean ring, free ring

Abstract: Let $m>n\geq 1$ be natural numbers such that m-n is odd; we prove that the identity $x^m = x^n$ implies $x^{m-n+1} = x$ in rings with unity. Moreover we describe the free ring corresponding to $x^n = x$, where $n=2^t$.

1. Preliminaries

During the last forty years the investigation of rings with polynomial identitities became a very important branch of ring theory. The

---

* Research (partially) supported by Hungarian National Foundation for Scientific Research grant no. 1813.

pioneering papers are due to Jacobson ([3], [4]). He proved that a ring satisfying $x^n = x$ $(n \geq 2)$ is commutative (in fact he proved a stronger version of this result). In the present note we introduce the notion of (m, n)-Boolean rings by generalizing Jacobson's above identity. The structure of (m, n)-Boolean rings heavily depends on the parity of the difference $m - n$. Our main result is a reduction theorem for the odd case. Another reduction theorem for the $x^n = x$ $(n \geq 2)$ case will be also stated. Finally, in the $n = 2^t$ case we describe the free ring statisfying $x^n = x$.

2. Reduction theorems for (m,n)-Boolean rings

Given two natural numbers $m > n \geq 1$, a ring R is said to be (m, n)-*Boolean* if $x^m = x^n$ for all $x \in R$.

Theorem 2.1. *Let R be an (m, n)-Boolean ring with unity, where $m-n$ is odd. Then R is $(m-n+1, 1)$-Boolean (and by Jacobson's well-known theorem we also get the commutativy of R).*

Proof. On applying $x^m = x^n$ to $x = -1_R$ we obtain $1_R + 1_R = 0$, i.e. that $2x = 0$ for all $x \in R$. Now we prove that R has no nilpotent element. Let $k \geq 2$ be an integer and suppose that $x^k = 0$ and $x^{k-1} \neq 0$ for a nilpotent $x \in R$. Using the binomial theorem, $(1_R + x^{k-1})^m = (1_R + x^{k-1})^n$ gives that $1_R + mx^{k-1} = 1_R + nx^{k-1}$, whence we get $(m - n)x^{k-1} = 0$. The odd parity of $m - n$ gives that $x^{k-1} = (m-n)x^{k-1} = 0$, a contradiction. The absence of nilpotent elements enables us to use a theorem of Andrunakievich and Rjabuhin (see [1]). According to this theorem R is a subdirect product of *domains* (i.e. not necessarily commutative rings without zero divisors) R_i $(i \in I)$. Since R_i is a factor of R, the identity $x^m = x^n$ remains true in R_i. But it can easily be seen that in a domain $x^m = x^n$ implies $x^{m-n+1} = x$. Hence any subdirect product of the rings R_i $(i \in I)$ will also satisfy $x^{m-n+1} = x$. ◇

Remark. In the case of even $m - n$ we cannot expect such a reduction theorem. For instance $\mathbf{Z}_{12}$ and the ring of 2×2 upper triangular matrices over a Boolean ring are examples of (4,2)-Boolean rings, the former has a nilpotent element and the latter is non-commutative.

Theorem 2.2. *An $(n,1)$-Boolean ring R is $(n^*,1)$-Boolean, where $n^*-1 = l.c.m.\{p^k-1|p$ is prime, p^k-1 is a divisor of $n-1\}$.*

Remark. The authors believe that this result is not essentially new, however we were not able to find a reference. Related investigations can be found in [2], [6] and [7].

Proof. We can proceed similarly to the proof of Th. 2.1. A domain satisfies $x^n = x$ if and only if it is a finite field of the form GF(p^k), where p^k-1 is a divisor of $n-1$. This result is explicit in [6] and in [5]. Since each subdirect factor R_i of R satisfies $x^{n^*} = x$, we get that their subdirect product R will also satisfy the same identity.◇

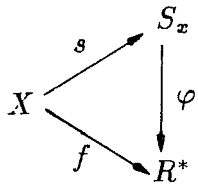
Remark. An immediate application of Th. 2.1. and Th. 2.2. can give the following reduction result. *Let R be a $(16,11)$-Boolean ring with unity, then* Th. 2.1. *gives* $(16,11) \Rightarrow (6,1)$, *and* Th. 2.2. *gives* $(6,1) \Rightarrow (2,1)$, *where* $2 = 6^*$. *Thus we get that R is a Boolean ring in the classical sense.*

3. The free $(2^t,1)$-Boolean ring

Theorem 3.1. *Let $n = 2^t$, then the free $(n,1)$-Boolean ring generated by a non-void set X can be obtained as the semigroup ring $\mathbb{Z}_2(S_x)$, where S_x is the free semigroup on X with defining relations $x^n = x$ and $xy = yx$.*

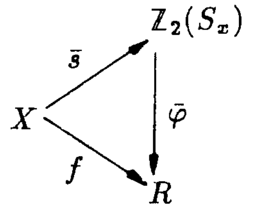
Proof. Using the polynomial theorem and the well known fact that polynomial coefficients of the form $\frac{n!}{i_1!i_2!\ldots i_k!}$ (where $n = 2^t = i_1 + i_2 + \ldots + i_k$ and $1 \le i_\nu \le n-1$ for some ν) are even integers, we obtain that $\mathbb{Z}_2(S_x)$ satisfies $x^n = x$.
In order to prove universality let $f : X \to R$ be a set mapping with R an $(n,1)$-Boolean ring. Since the multiplicative semigroup R^* of R satisfies $x^n = x$ and $xy = yx$ (by Jacobson's theorem) there is unique semigroup-homomorphic extension φ of f making the diagram (3.1) commute

(3.1)



Now it is easy to see that the definiton $\bar{\varphi}(\sum_{\sigma \in S_x} \bar{n}_\sigma \sigma) = \sum_{\sigma \in S_x} n_\sigma \varphi(\sigma)$ with $\bar{n}_\sigma = n_\sigma + (2) \in \mathbf{Z}_2$ is correct and gives a $\mathbf{Z}_2(S_x) \to R$ ring-homomorphism making (3.2) commute (we need $2R = 0$!)

(3.2)



$\bar{s}(x) = 1.s(x)$

Since the subset $\bar{s}(X) \subseteq \mathbf{Z}_2(S_x)$ generates $\mathbf{Z}_2(S_x)$ as a ring, the unicity of $\bar{\varphi}$ is clear. $\diamondsuit$

References

[1] ANDRUNAKIEVICH, V. and RJABUHIN, J.M.: Rings without nilpotent elements, and completely prime ideals, *Doklady Akad. Nauk USSR* **180** (1968), 9 – 11.

[2] AYOUB, R. and AYOUB, C.: On the commutativity of rings, *Amer. Math. Monthly* **71** (1964), 267 – 271.

[3] JACOBSON, N.: Structure theory for algebraic algebras of bounded degree, *Ann. of Math.* **46** (1945), 695 – 707.

[4] JACOBSON, N.: Structure of rings, *Amer. Math. Soc. Colloq. Publ.* **37**, Providence, 1964.

[5] VAN LEEUWEN, L.C.A.: A note on primitive classes of arithmetic rings, *Publs. Math.* (Debrecen) **23** (1976), 41 – 43.

[6] LUH, J.: On the structure of J-rings, *Amer. Math. Monthly* **74** (1976), 163 – 166.

[7] MICHLER, G. and WILLE, R.: Die primitiven Klasse arithmetischer Ringe, *Math. Z.* **113** (1979), 369 – 372.