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ON THE DUAL SPACE OF AN MS-ALGEBRA

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Abstract: We provide a characterisation of all subvarieties of the variety **MS** of MS-algebras via their dual spaces. It consists of universal sentences in disjunctive normal form which involve only one variable. We apply this result to the construction of distributive lattices on which there can be defined (up to isomorphism) a unique MS-algebra which belongs to a preassigned class.

In 1983 we introduced the notion of an MS-algebra as a common abstraction of a de Morgan algebra and a Stone algebra [3]. Precisely, an MS-algebra is a bounded distributive lattice L endowed with a unary operation $a \to a^\circ$ such that

$$(\forall a \in L) a \leq a^{\circ\circ};$$

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$$(\forall a, b \in L)(a \wedge b)^\circ = a^\circ \vee b^\circ;$$
$$1^\circ = 0.$$

Clearly, an MS-algebra is a distributive Ockham algebra ([2], [6] and [9]).

The class **MS** of MS-algebras is equational and all its subclasses were described in [4] by identities that involve at most two variables. We keep the numbering which was adopted in [4, page 159].

R. Beazer [1] and ourselves [5] showed the role that duality theory can play in the study of **MS**. Throughout we assume familiarity with H.A. Priestley's topological duality for bounded distributive lattices as it is presented in [8]. We only recall the facts we need.

A *Priestly space* X is a compact totally order disconnected space, the property of *total order disconnectedness* being defined as follows:
(TOD) given $x \not\leq y$ in X, there exists a clopen order ideal $V \subseteq X$ such that $x \notin V$ and $y \in V$.

The lattice of clopen order ideals of X is denoted by $\mathcal{O}(X)$ and is isomorphic to L_X, the dual algebra of X. In any Priestley space X, for each $x \in X$ there is $y \leq x$ such that y is minimal with respect to the partial order. The set of all minimal points of X is denoted by $\min X$.

Since MS-algebras are bounded distributive lattices, they are dually equivalent to some suitable category of Priestley spaces. In fact, an *MS-space* X is a Priestley space endowed with a continuous order reversing map $g : X \to X$ which satisfies

$$(\forall x \in X) x \geq g^2(x).$$

In [5] we observed that the latter condition implies

$$(\forall x \in X) g^3(x) = g(x)$$

and that to determine such a mapping g it suffices to find a closed subspace X_1 of X which possesses a dual order isomorphism h, then a decreasing order preserving retraction $f : X \to X_1$, and to take $g = h \circ f$. Clearly, $g^2(x) = x$ if and only if $x \in X_1$. It follows that $X_1 \supseteq \min X$.

The unary operation $^\circ$ on L_X is defined by

$$\begin{aligned} I^\circ &= X \setminus g^{-1}(I) \qquad (I \in \mathcal{O}(X)) \\ &= \{x \in X : g(x) \notin I). \end{aligned}$$

Consequently,

$$I^{\circ\circ} = \{x \in X : g^2(x) \in I\} \supseteq I.$$

We use the symbols $\gtrless$ and $\|$ to indicate that two elements are comparable and incomparable respectively. The signs $\subseteq$ and $\subset$ are employed for inclusion and strict inclusion respectively. The expression "L properly belongs to **X**" means that **X** is the least subvariety of **MS** to which L belongs.

After establishing the main result of this paper, that is, the characterisation of all the subvarieties of **MS** via their dual spaces, we formulate some direct consequences which highlight the crucial rôle played by duality theory. The characterisation theorem is then used to solve the following problem: given a subvariety **X** of **MS**, how to construct a distributive lattice on which there can be defined a unique **MS**-algebra which belongs to **X**. We show that this is possible except for the class **S** of Stone algebras.

Theorem 1. *Let* $(L;^{\circ})$ *be an* MS-*algebra and* $(X_L; g)$ *its dual space. Then* $(L;^{\circ})$ *satisfies the identity on the left if and only if* $(X_L; g)$ *satisfies the corresponding formula on the right:*

(2)	$a \vee a^{\circ} = 1$	(II)	$x = g(x)$
(2_d)	$a \wedge a^{\circ} = 0$	(II_d)	$g(x) = g^2(x)$
(3)	$a = a^{\circ\circ}$	(III)	$x = g^2(x)$
(4)	$a \wedge a^{\circ} = a^{\circ\circ} \wedge a^{\circ}$	(IV)	$x = g^2(x)$ or $x > g(x)$
(4_d)	$a \vee a^{\circ} = a^{\circ\circ} \vee a^{\circ}$	(IV_d)	$x = g^2(x)$ or $x < g(x)$
(5)	$(a \wedge a^{\circ}) \vee b \vee b^{\circ} = b \vee b^{\circ}$	(V)	$x \gtrless g(x)$
(6)	$(a \wedge a^{\circ}) \vee b^{\circ\circ} \vee b^{\circ} = b^{\circ\circ} \vee b^{\circ}$	(VI)	$g(x) \gtrless g^2(x)$
(7)	$(a \wedge a^{\circ}) \vee b \vee b^{\circ} = (a^{\circ\circ} \wedge a^{\circ}) \vee b \vee b^{\circ}$	(VII)	$x = g^2(x)$ or $x \lessgtr g(x)$
(8)	$a \vee b^{\circ} \vee b^{\circ\circ} = a^{\circ\circ} \vee b^{\circ} \vee b^{\circ\circ}$	(VIII)	$x = g^2(x)$ or $g^2(x) \leq g(x)$
(9)	$(a \wedge a^{\circ}) \vee b^{\circ} \vee b^{\circ\circ} = (a^{\circ\circ} \wedge a^{\circ}) \vee b^{\circ} \vee b^{\circ\circ}$	(IX)	$x = g^2(x)$ or $g(x) \gtrless g^2(x)$.

Proof. (2) $\Leftrightarrow$ (II).

$$\begin{aligned}(2) &\Leftrightarrow I \cup I^{\circ} = X && (\forall I \in \mathcal{O}(X))\\ &\Leftrightarrow x \in I \text{ or } g(x) \notin I && (\forall I \in \mathcal{O}(X))\\ &\Leftrightarrow x \notin I \text{ implies } g(x) \notin I && (\forall I \in \mathcal{O}(X)).\end{aligned}$$

Let (2) be satisfied. Then if $x \neq g(x)$, by (TOD) there is $V \in \mathcal{O}(X)$ which separates the elements x and $g(x)$, which constradicts the last equivalence. The converse is straightforward. $\diamondsuit$

Observe that (II) implies that X is an antichain. In fact, if $y \geq x$ then

$y = g(y) \le g(x) = x$, hence $x = y$.

$(2_d) \Leftrightarrow (\mathrm{II}_d)$.

$$\begin{array}{llr} (2_d) & \Leftrightarrow I \cap I^\circ = \emptyset & (\forall I \in \mathcal{O}(X)) \\ & \Leftrightarrow \{x \in X : x \in I \text{ and } g(x) \notin I\} = \emptyset & (\forall I \in \mathcal{O}(X)) \\ & \Leftrightarrow x \in I \text{ implies } g(x) \in I & (\forall I \in \mathcal{O}(X)) \\ & \Leftrightarrow g(x) \le x & (\forall x \in X) \text{ by (TOD)} \\ & \Leftrightarrow g(x) = g^2(x) & (\forall x \in X). \end{array}$$

Indeed, if $g(x) \le x$ identically, then $g^2(x) \le g(x)$ but also $g^2(x) \ge g(x)$ since g is order reversing. It follows that $g(x) = g^2(x)$. The other direction is clear since $g^2(x) \le x$ always. $\Diamond$

The condition (II_d) is equivalent to

$(\mathrm{II'}_d)$ every connected component A of X contains exactly one element a of $\min X$ and $g(A) = \{a\}$.

Clearly $(\mathrm{II'}_d)$ implies (II_d). Conversely, suppose that (II_d) holds (i.e. $g = g^2$). Since g is order reversing and g^2 is order preserving, it follows from $x \gtrless y$ that $g(x) = g(y)$. Hence, since A is connected, $g(A)$ is a singleton, necessarily a minimal element, and $(\mathrm{II'}_d)$ is verified. $\Diamond$

Note also that $(\mathrm{II'}_d)$ has as direct consequence the well-known fact that in a Stone algebra every prime ideal contains exactly one minimal prime ideal.

(3) $\Leftrightarrow$ (III).

$$\begin{array}{llr} (3) & \Leftrightarrow I = I^{\circ\circ} & (\forall I \in \mathcal{O}(X)) \\ & \Leftrightarrow I^{\circ\circ} \subseteq I & (\forall I \in \mathcal{O}(X)) \\ & \Leftrightarrow g^2(x) \in I \text{ implies } x \in I & (\forall I \in \mathcal{O}(X)) \\ & \Leftrightarrow g^2(x) = x & (\forall x \in X). \end{array}$$

The last equivalence is justified as follows: if $x > g^2(x)$, then by (TOD) there is $V \in \mathcal{O}(X)$ such that $g^2(x) \in V$ and $x \notin V$, a contradiction. $\Diamond$

(4) $\Leftrightarrow$ (IV).

$$\begin{array}{llr} (4) & \Leftrightarrow I \cap I^\circ = I^{\circ\circ} \cap I^\circ & (\forall I \in \mathcal{O}(X)) \\ & \Leftrightarrow I^{\circ\circ} \cap I^\circ \subseteq I & (\forall I \in \mathcal{O}(X)) \\ & \Leftrightarrow (g^2(x) \in I \text{ and } g(x) \notin I) \text{ implies } x \in I & (\forall I \in \mathcal{O}(X)). \end{array}$$

Let (4) be satisfied. If $x > g^2(x)$ and $g(x) \not\le x$, then by (TOD) there exist $V, W \in \mathcal{O}(X)$ such that $x \in V$, $g(x) \notin V$, $g^2(x) \in W$, $x \notin W$.

We thus have $g^2(x) \in V \cap W$ and $g(x) \notin V \cap W$ whereas $x \notin V \cap W$, contradicting (4). It follows that $x > g(x)$.

Conversely, let (IV) be satisfied. If $g^2(x) \in I$ and $g(x) \notin I$ for some $I \in \mathcal{O}(X)$, then $g(x) \not\leq g^2(x)$, hence $g(x) \not\leq x$, and, by (IV), $x = g^2(x)$, $x \in I$ and (4) holds. $\diamondsuit$

<u>$(4_d) \Leftrightarrow (\mathrm{IV}_d)$.</u>

$$\begin{aligned}
(4_d) \quad &\Leftrightarrow I^\circ \cup I^{\circ\circ} \subseteq I^\circ \cup I && (\forall I \in \mathcal{O}(X))\\
&\Leftrightarrow I^{\circ\circ} \subseteq I^\circ \cup I && (\forall I \in \mathcal{O}(X))\\
&\Leftrightarrow g^2(x) \in I \text{ implies } (x \in I \text{ or } g(x) \notin I) && (\forall I \in \mathcal{O}(X))\\
&\Leftrightarrow (x \notin I \text{ and } g(x) \in I) \text{ implies } g^2(x) \notin I && (\forall I \in \mathcal{O}(X)).
\end{aligned}$$

Let (4_d) be satisfied and $x > g^2(x)$. If $x = g(x)$ then $x = g^2(x)$, contradicition. If $x \not\leq g(x)$, then there is $V \in \mathcal{O}(X)$ such that $g(x) \in V$ and $x \notin V$. Since by assumption $x \not\leq g^2(x)$, there is $W \in \mathcal{O}(X)$ such that $g^2(x) \in W$ and $x \notin W$. Thus we have $x \notin V \cup W$, $g(x) \in V \cup W$ and nevertheless $g^2(x) \in V \cup W$.

Conversely, let (IV_d) be satisfied. If $x = g^2(x)$ then (4_d) is trivially satisfied. If $g(x) > x$, then every order ideal which contains $g(x)$ contains x as well and (4_d) holds. $\diamondsuit$

<u>(5) $\Leftrightarrow$ (V).</u>

$$\begin{aligned}
(5) \quad \Leftrightarrow \quad & I \cap I^\circ \subseteq J \cup J^\circ && (\forall I, J \in \mathcal{O}(X))\\
\Leftrightarrow \quad & (x \in I \text{ and } g(x) \notin I) \text{ implies} \\
& (x \in J \text{ or } g(x) \notin J) && (\forall I, J \in \mathcal{O}(X)).
\end{aligned}$$

Let (5) be satisfied. If $g(x)||x$, then there are $V, W \in \mathcal{O}(X)$ such that $x \in V$, $g(x) \notin V$, $g(x) \in W$ and $x \notin W$, contradiciting the preceding implication.

Now suppose that (V) is satisfied. The case $g(x) \leq x$ is straightforward. Now if $g(x) > x$, any decreasing subset which does not contain x does not contain $g(x)$ either. $\diamondsuit$

<u>(6) $\Leftrightarrow$ (VI).</u>

$$\begin{aligned}
(6) \quad \Leftrightarrow \quad & I \cap I^\circ \subseteq J^\circ \cup J^{\circ\circ} && (\forall I, J \in \mathcal{O}(X))\\
\Leftrightarrow \quad & I^{\circ\circ} \cap I^\circ \subseteq J^\circ \cup J^{\circ\circ} && (\forall I, J \in \mathcal{O}(X))\\
\Leftrightarrow \quad & (g^2(x) \in I \text{ and } g(x) \notin I) \text{ implies} \\
& (g^2(x) \in J \text{ or } g(x) \notin J) && (\forall I, J \in \mathcal{O}(X)).
\end{aligned}$$

The proof is similar to the preceding one, just changing x into $g^2(x)$. $\Diamond$

(7) $\Leftrightarrow$ (VII).

$$\begin{aligned}(7) &\Leftrightarrow (I^{\circ\circ} \cap I^{\circ}) \cup J \cup J^{\circ} \subseteq (I \cap I^{\circ}) \cup J \cup J^{\circ} && (\forall I, J \in \mathcal{O}(X))\\ &\Leftrightarrow I^{\circ\circ} \cap I^{\circ} \subseteq I \cup J \cup J^{\circ} && (\forall I, J \in \mathcal{O}(X))\\ &\Leftrightarrow (g^2(x) \in I \text{ and } g(x) \notin I) \text{ implies}\\ &\quad (x \in I \text{ or } x \in J \text{ or } g(x) \notin J) && (\forall I, J \in \mathcal{O}(X))\\ &\Leftrightarrow x \text{ satisfies (IV) or (V)} && (\forall x \in X)\\ &\Leftrightarrow x = g^2(x) \text{ or } x \gtrless g(x). \quad \Diamond\end{aligned}$$

(8) $\Leftrightarrow$ (VIII).

$$\begin{aligned}(8) &\Leftrightarrow I^{\circ\circ} \cup J^{\circ} \cup J^{\circ\circ} \subseteq I \cup J^{\circ} \cup J^{\circ\circ} && (\forall I, J \in \mathcal{O}(X))\\ &\Leftrightarrow I^{\circ\circ} \subseteq I \cup J^{\circ} \cup J^{\circ\circ} && (\forall I, J \in \mathcal{O}(X))\\ &\Leftrightarrow g^2(x) \in I \text{ implies}\\ &\quad (x \in I \text{ or } g^2(x) \in J \text{ or } g(x) \notin J) && (\forall I, J \in \mathcal{O}(X))\\ &\Leftrightarrow (g^2(x) \in I \text{ and } x \notin I) \text{ implies}\\ &\quad (g^2(x)s \in J \text{ or } g(x) \notin J) && (\forall I, J \in \mathcal{O}(X))\end{aligned}$$

Let (8) be satisfied and $x > g^2(x)$. If $g^2(x) \not\leq g(x)$, then there is $V \in \mathcal{O}(X)$ such that $g(x) \in V$ and $g^2(x) \notin V$, which contradicts (8).

Conversely, let (VIII) be satisfied. Since every decreasing subset which does not contain $g^2(x)$ does not contain $g(x)$ either, (8) is satisfied. $\Diamond$

(9) $\Leftrightarrow$ (IX).

The proof goes along the same lines as in (7) $\Leftrightarrow$ (VII).

$$\begin{aligned}(9) &\Leftrightarrow x \text{ satisfies (IV) or (VI)} && (\forall x \in X)\\ &\Leftrightarrow x = g^2(x) \text{ or } g(x) \gtrless g^2(x). \quad \Diamond\end{aligned}$$

Corollary 1. *All the subvarieties of* **MS** *can be characterised via their spaces by the disjunction of at most three universal sentences which involve only one variable.*

The results are recorded in the subvariety lattice represented on the page 102. Note that g° means id_X.

Proof. Theorem 1 yields the characterisation of the subvarieties which are defined by a unique identity. The other non-trivial subvarieties are characterised by the conjunction of two or three conditions. An easy computation provides the corresponding conditions on the dual space in disjunctive form. For instance, $L \in \mathbf{S} \vee \mathbf{K}$ if and only if $(X_L; g)$ satisfies (IV),(V) and (VIII). The conjunction of (IV) and (VIII) is equivalent to the disjunction of $x = g^2(x)$ and $x > g(x) \geq g^2(x)$, which in turn is equivalent to the disjunction of $x = g^2(x)$ and $g(x) = g^2(x)$. Finally, the conjunction of (IV), (V) and (VIII) is equivalent to the disjunction of $x = g^2(x) \gtrless g(x)$ and $g(x) = g^2(x)$. $\diamondsuit$

Corollary 2.- *If X has at least two connected components A, B and $g(A) \subseteq B$, then L does not satisfy* (6). *If moreover $g(A) \subset B$, then L properly belongs to $\mathbf{M}_1$.*

Proof. The first part is obvious. As for the second part, observe that there is $x \in B \setminus g(A)$ such that $x \neq g^2(x)$ and $g(x) || g^2(x)$, hence L does not satisfy (9). $\diamondsuit$

Corollary 3. *If $X_1 \subset X$ and X_1 is convex, then L does not satisfy* (4_d).

Proof. If L satisifies (4_d) but not (3), then there is $x \in X \setminus X_1$ such that $g^2(x) < x < g(x)$ and X_1 is not convex. $\diamondsuit$

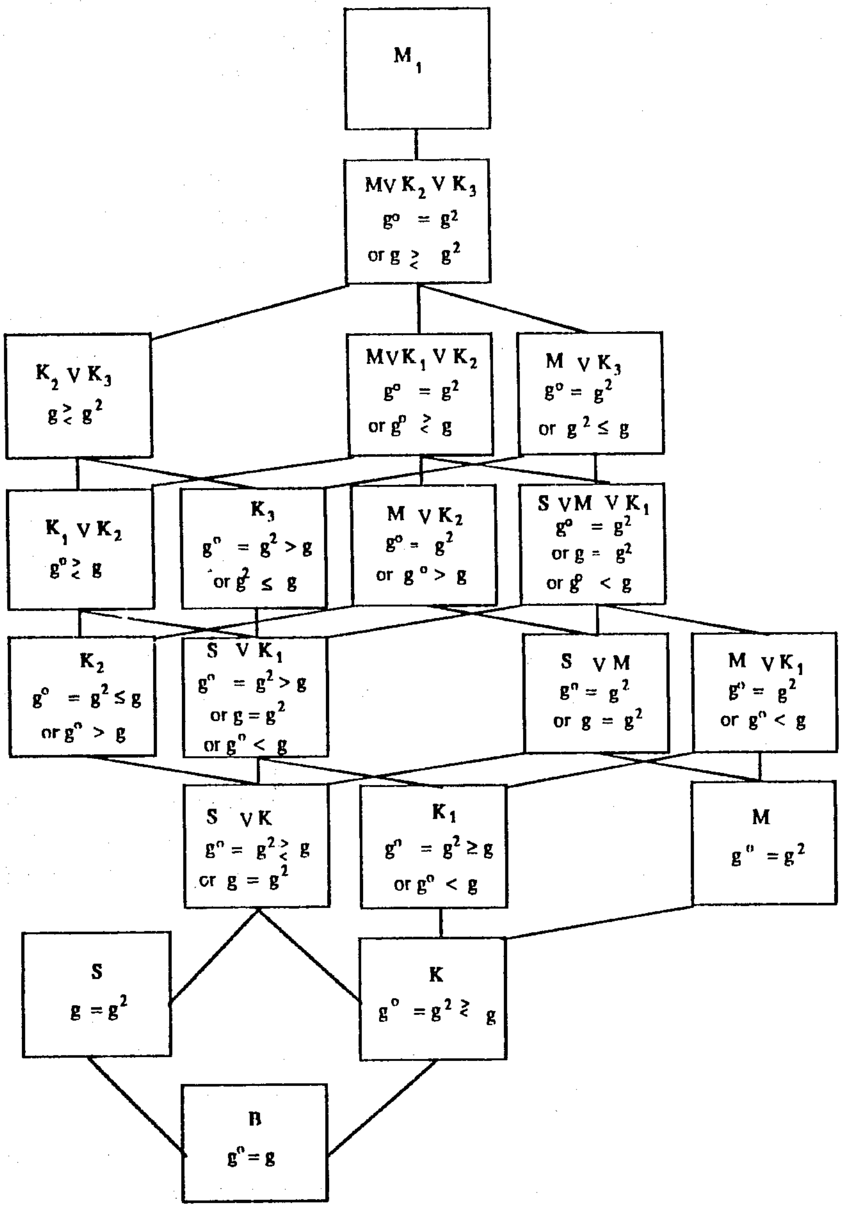
Corollary 4. *If X_1 is open and convex, then the dual of X_1 is a principal ideal of L the generator of which is the least element a of L such that $a^\circ = 0$.*

Proof. Since X_1 is always closed, it is clopen; since it contains $\min X$ and is assumed to be convex, it is decreasing. Its dual is a principal ideal $a^\downarrow$ of L. By its very definition $X_1^\circ = \emptyset$. Let $Y \in \mathcal{O}(X)$. Clearly $Y \supseteq X_1$ if and only if $Y^\circ = \emptyset$. So the least element Y of $\mathcal{O}(X)$ such that $Y^\circ = \emptyset$ is X_1. $\diamondsuit$

We already noticed that (III) is equivalent to $X = X_1$, and that (II) implies $X = X_1$. Can the verification of some of the other axioms be restricted to X_1? The answer is affirmative as shown by

Corollary 5. *The axioms* (II_d) *and* (VI) *are satisfied if (and only if) they are so on X_1. If X_1 is totally ordered, then L satisfies* (6).

Proof. Just observe that $X_1 = \{g(x) : x \in X\} = \{g^2(x) : x \in X\}$. $\diamondsuit$


M_1
$M \vee K_2 \vee K_3$
$g^o = g^2$
or $g \gtrless g^2$
$K_2 \vee K_3$
$g \gtrless g^2$
$M \vee K_1 \vee K_2$
$g^o = g^2$
or $g^o \gtrless g$
$M \vee K_3$
$g^o = g^2$
or $g^2 \leq g$
$K_1 \vee K_2$
$g^o \gtrless g$
K_3
$g^o = g^2 > g$
or $g^2 \leq g$
$M \vee K_2$
$g^o = g^2$
or $g^o > g$
$S \vee M \vee K_1$
$g^o = g^2$
or $g = g^2$
or $g^o < g$
K_2
$g^o = g^2 \leq g$
or $g^o > g$
$S \vee K_1$
$g^o = g^2 > g$
or $g = g^2$
or $g^o < g$
$S \vee M$
$g^o = g^2$
or $g = g^2$
$M \vee K_1$
$g^o = g^2$
or $g^o < g$
$S \vee K$
$g^o = g^2 \gtrless g$
or $g = g^2$
K_1
$g^o = g^2 \geq g$
or $g^o < g$
M
$g^o = g^2$
S
$g = g^2$
K
$g^o = g^2 \gtrless g$
B
$g^o = g$

There is a significant difference between Stone algebras and de Morgan algebras: when a bounded distributive lattice can be made into a Stone algebra, this can be done in only one way (in other words, the lattice structure determines the unary operation $^\circ$); on the contrary, many bounded distributive lattices admit various definitions of the unary operation $^\circ$ which satisfy the axioms of a de Morgan algebra (more generally, of an MS-algebra of a given class other than **S** or **B**). This provokes the question as to whether we can find, for a given subvariety **X** of MS-algebras, a distributive lattice L on which there can be defined to within isomorphism a *unique* MS-algebra structure such that $(L,^\circ) \in \mathbf{X}$. A subvariety **X** for which this is the case will be called *saturated.*

Theorem 2. *All subvarieties of MS-algebras, other than* **S**, *are saturated.*

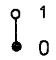
Proof. We first show that the subvariety **S** of Stone algebras is not satureted. Suppose that L is a distributive lattice on which there can be defined an MS-algebra structure such that $(L,^\circ)$ belongs properly to **S**. Let $(X_L; g)$ be the corresponding MS-space. We have $X_L = (A_i)_{i\in I}$ where the A_i are the connected components of X_L. By (II'$_d$) every A_i has a least element a_i. Moreover, not every A_i consists of a singleton, for otherwise $(L,^\circ) \in \mathbf{B}$. For a given A_i that is not a singleton, say A_{i*}, choose an element $x_{i*} \neq a_{i*}$ and take $X_1 = \{a_i : i \in I\} \cup \{x_{i*}\}$. Since $a_{i*} = g(x_{i*}) \neq g^2(x_{i*}) = x_{i*}$,we obtain an MS-algebra that does not belong to **S**.

Now, for each non-trivial subvariety **X**, other than **S**, we give an example of a distributive lattice L that can be made into an MS-algebra in only one way with $(L,^\circ) \in \mathbf{X}$. This we achieve by considering in each case an appropriate MS-space. In all examples the space X is connected and, except for the class **B** of Boolean algebras, $|\min X| \geq 2$ since otherwise X_1 could be chosen in various ways. The black circles correspond to the elements of X which do not belong to X_1, and to the meet-irreducible elements of L_X other than 1.

B

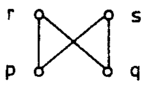
o p

x	p
g(x)	p

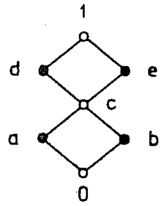


α	0	1
α^0	1	0

K

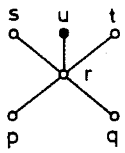


x	p	q	r	s
g(x)	r	s	p	q

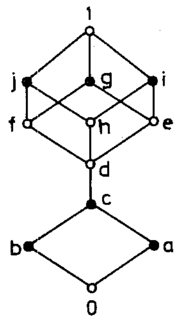


α	0	a	b	c	d	e	1
α^0	1	e	d	c	b	a	0

S ∨ K

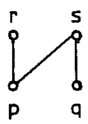


x	p	q	r	s	t	u
q(x)	s	t	r	p	q	r

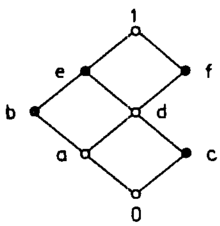


α	0	a	b	c	d	e	f	g	h	i	j	1
α^0	1	j	i	h	c	b	a	0	c	b	a	0

M

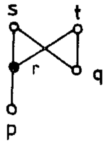


x	p	q	r	s
g(x)	s	r	q	p

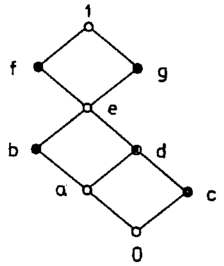


α	0	a	b	c	d	e	f	1
α^0	1	e	b	f	d	a	c	0

K_1

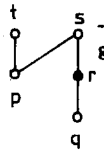


x	p	q	r	s	t
g(x)	s	t	s	p	q

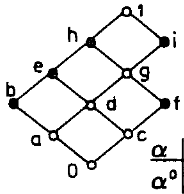


α	0	a	b	c	d	e	f	g	1
α^0	1	g	g	f	e	e	c	b	0

M_1

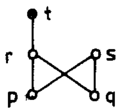


x	p	q	r	s	t
g(x)	s	t	t	p	q

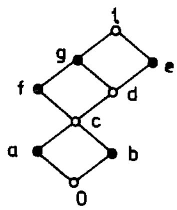


α	0	a	b	c	d	e	f	g	h	i	1
α^0	1	h	b	i	g	a	i	g	a	f	0

K_2

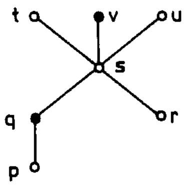


x	p	q	r	s	t
g(x)	r	s	p	q	p

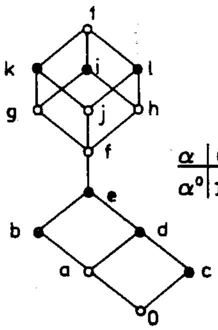


α	0	a	b	c	d	e	f	g	1
α^0	1	f	e	c	b	b	a	0	0

$S \vee K_1$

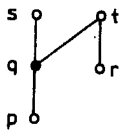


x	p	q	r	s	t	u	v
g(x)	t	t	u	s	p	r	s

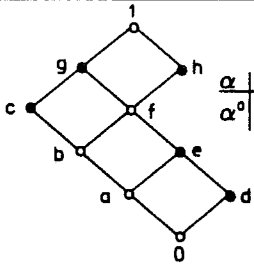


α	0	a	b	c	d	e	f	g	h	i	j	k	l	1
α^0	1	1	1	k	j	j	e	c	b	0	e	c	b	0

$M \vee K_1$

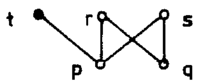


x	p	q	r	s	t
g(x)	t	t	s	r	p

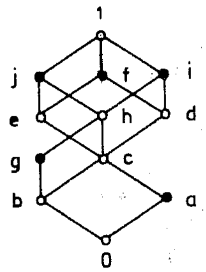


α	0	a	b	c	d	e	f	g	h	1
α^0	1	g	g	c	h	f	f	b	d	0

K_3

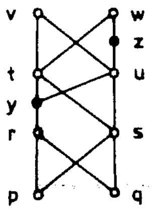


x	p	q	r	s	t
g(x)	r	s	p	q	r

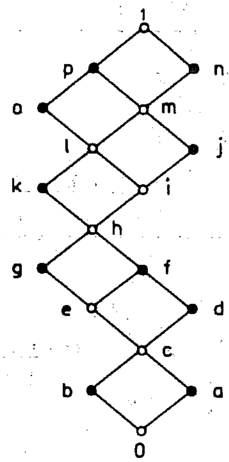


α	0	a	b	c	d	e	f	g	h	i	j	1
α^0	1	j	i	h	g	a	0	i	h	g	a	0

$K_1 \vee K_2$



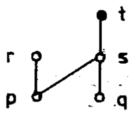
x	p	q	r	s	t	u	v	w	y	z
g(x)	v	w	t	u	r	s	p	q	t	s



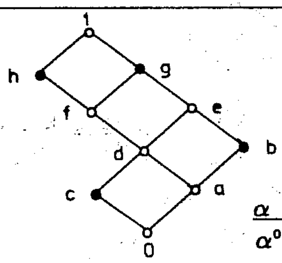
α	0	a	b	c	d	e	f	g	h	i	j
α^0	1	p	n	m	k	j	h	j	h	g	g

α	k	l	m	n	o	p	1
α^0	d	c	c	b	a	a	0

$M \vee K_2$

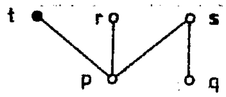


x	p	q	r	s	t
g(x)	s	r	q	p	p

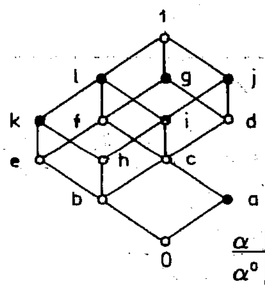


α	0	a	b	c	d	e	f	g	h	1
α^0	1	e	b	h	d	a	c	0	c	0

$M \vee K_3$

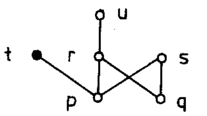


x	p	q	r	s	t
g(x)	s	r	q	p	s

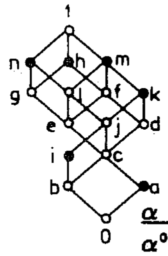


α	0	a	b	c	d	e	f	g	h	i	j	k	l	1
α^0	1	j	l	i	a	k	h	0	l	i	a	k	h	0

$K_2 \vee K_3$

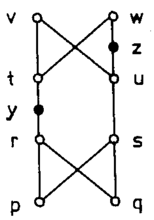


x	p	q	r	s	t	u
g(x)	r	s	p	q	r	p

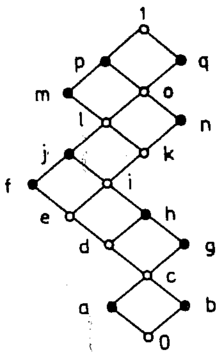


α	0	a	b	c	d	e	f	g	h	i	j	k	l	m	n	1
α^0	1	n	k	j	i	a	0	a	0	k	j	i	a	0	a	0

$M \vee K_1 \vee K_2$

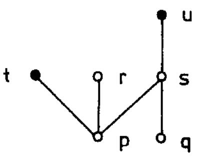


x	p	q	r	s	t	u	v	w	y	z
g(x)	w	v	u	t	s	r	q	p	u	r

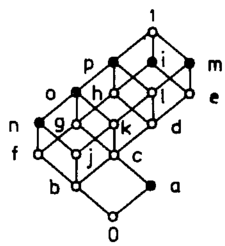


α	0	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	1
α^0	1	p	q	0	j	j	f	n	i	i	e	g	c	a	g	c	a	b	0

$M \vee K_2 \vee K_3$

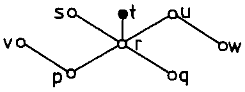


x	p	q	r	s	t	u
g(x)	s	r	q	p	s	p

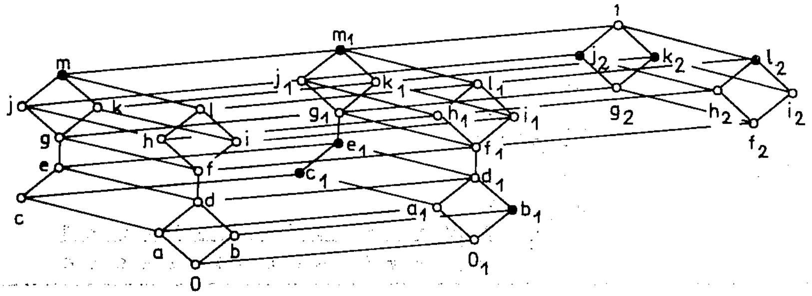


α	0	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	1
α^0	1	i	o	g	a	a	o	g	a	a	n	f	0	0	n	f	0	0

S ∨ M



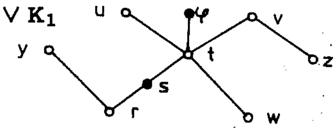
x	p	q	r	s	t	u	v	w
g(x)	u	s	r	q	r	p	w	v



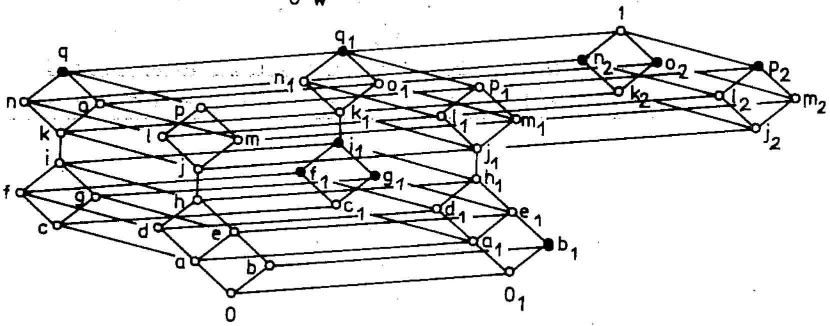
α	0	a	b	c	d	e	f	g	h	i	j	k	l	m	0_1	a_1	b_1	c_1	d_1	e_1	f_1	g_1	h_1	i_1	j_1
α^0	1	m_1	k_2	m	k_1	k	e_1	e	c_1	e_1	c	e	c_1	c	l_2	l_1	i_2	l	i_1	i	d_1	d	a_1	d_1	a

α	k_1	l_1	m_1	f_2	g_2	h_2	i_2	j_2	k_2	l_2	1
α^0	d	a_1	a	b_1	b	0_1	b_1	0	b	0_1	0

S ∨ M ∨ K_1



x	r	s	t	u	v	w	y	z	φ
g(x)	v	v	t	w	r	u	z	y	t



α	0	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	0_1	a_1	b_1	c_1	d_1	e_1	f_1
α^0	1	q_1	o_2	q	q_1	o_1	q	o	o_1	o	i_1	i	f_1	i_1	f	f_1	i	f	p_2	p_1	m_2	p	p_1	m_1	p

α	g_1	h_1	i_1	j_1	k_1	l_1	m_1	n_1	o_1	p_1	q_1	j_2	k_2	l_2	m_2	n_2	o_2	p_2	1
α^0	m	m_1	m	h_1	h	d_1	h_1	d	h	d_1	d	b_1	b	0_1	b_1	0	b	0_1	0

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