

SEMIRINGS WITHOUT ZERO DIVISORS

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Abstract: The main subject of this paper are semirings with an absorbing zero 0 which are zero divisor free (ZDF), but which have zero sums. We show that each such semiring S contains a greatest subring $R \supset \{0\}$ (in the usual meaning, even if $(S, +)$ is not commutative) which has no α -fiers and of course no zero divisors. Conversely, each ring R of this kind occurs as the greatest subring of some semiring S as above, where S itself is not a ring. In this situation, various structural results on S , R and $U = S \setminus R \neq \emptyset$ are proved, e.g. that each $s \neq 0$ in S has infinite additive order. We also deal with semirings which are multiplicatively left or right cancellative or even both (briefly MLC, MRC and MC). For a semiring S with zero, each of these assumptions implies that S is ZDF, but not conversely. We show that each semiring with zero sums is MLC iff it is MRC and thus MC. Moreover, such a semiring S has an absorbing zero, $(S, +)$ is commutative and cancellative, and S is embeddable into a ring which is also MC. Finally, we prove by examples that all our results on proper ZDF semirings S with an absorbing

zero and zero sums are fairly complete. In particular, each ring R satisfying the necessary conditions above can be embedded into such a semiring S which is MC as well as into one which is merely ZDF.

1. Introduction

A (2,2)-algebra $S = (S, +, \cdot)$ is called a *semiring* iff $(S, +)$ and (S, \cdot) are arbitrary semigroups, which are connected by ring-like distributivity, i.e. $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ hold for all $a, b, c \in S$. This rather general concept has been investigated in several papers (e.g. [3], [5], [10], [14], [16], [18], [20], [21]), whereas all semirings occurring in various applications in the last two decades, in particular in different branches of Theoretical Computer Science (cf. e.g. [1], [2], [7], [12], [13], [15]), have commutative addition. Moreover, they mostly have a zero, which is then always assumed to be absorbing (cf. Section 2). Our main purpose is of course to add some knowledge on semirings of the latter kind, but we do not assume that $(S, +)$ is commutative for all results which are in fact independent of this assumption. We also say explicitly if a zero is assumed to be absorbing. We further call a semiring S *non-trivial* iff it contains at least two elements, and *proper* iff S is not a ring.

Let S be a semiring with a zero o . Then S is called *zero divisor free* (briefly ZDF) iff $ab = o$ implies $a = o$ or $b = o$ for all $a, b \in S$. Now either $a + b = o$ implies $a = b = o$ for all $a, b \in S$, or there is at least one pair $(r, s) \in S^* \times S^*$ for $S^* = S \setminus \{o\}$ satisfying $r + s = o$, called a *zero sum* of S . In the first case, for a non-trivial ZDF semiring, $(S^*, +, \cdot)$ is a subsemiring of $(S, +, \cdot)$, and by Lemma 2.1 any semiring T occurs as such a subsemiring S^* of a ZDF semiring S . So the assumption that a semiring S is ZDF provides in this case no more results than those which concern any semiring S^* .

Therefore our interest is with the second case, and the subject of this paper are ZDF semirings which have zero sums, in particular those where the zero is absorbing. We prepare these investigation by some

general concepts and statements on semirings in Section 2.

Then we show that a ZDF semiring S with an absorbing zero o has zero sums iff it contains a nontrivial subring R' with o as zero, in the usual meaning that $(R', +)$ is commutative even if $(S, +)$ is not. Moreover, the greatest subring R of this kind consists of all elements r, s, \dots of S which occur in zero sums of S (cf. Thm. 3.3). Provided that S itself is not a ring, various structural results on S , R and $U = S \setminus R$ and their interrelation are obtained in Section 3. We only mention here that each element $s \neq o$ of S has infinite additive order, that R is a ring which has no α -fier for any $\alpha \in \mathbb{N}$ as defined in [6] (cf. Section 2 and Thm. 3.7), and that (under a rather general supplementary assumption) S is an Everett-Rédei semiring extension of R as introduced in [14] (cf. Suppl. 3.6).

In the following section we sharpen some of the above results, dealing with semirings which are multiplicatively left or right cancellative or even both (briefly MLC, MRC and MC, cf. Section 2). For a semiring S with a zero, each of these assumptions implies that S is ZDF, but not conversely, and there are even semirings which are e.g. MLC and do not contain any multiplicatively right cancellable element. So it is surprising that each semiring S with zero sums is MLC iff it is MRC and hence MC, which also yields that the zero of S is absorbing and that $(S, +)$ is cancellative and commutative (cf. Thm. 4.1). We further obtain that such a MC semiring S with zero sums is embeddable into a ring and the smallest ring $D(S)$ of this kind is also MC (cf. Thm. 4.4).

In Section 5 we recall that a well-known construction to embed rings in those with an identity transfers similarly to semirings (cf. Prop. 5.1). Using this and results of Section 3, we show that each ZDF semiring S with commutative addition and an absorbing zero is embeddable into a semiring of the same kind which has an identity (cf. Thm. 5.3 and Remark 5.5). Moreover, the constructions mentioned above are also basic for Thm. 5.6 and Constr. 6.1, which leads to Thm. 6.2. The purpose of these theorems will be explained in the following.

Concerning the completeness of our structural statements on proper ZDF or MC semirings S with an absorbing zero and zero sums, the greatest subring R of such a semiring S is of course also ZDF and, as

mentioned above, has no α -fiers. Conversely, each ring R of this kind occurs as the greatest subring of such a semiring S . In fact, we give two general constructions for those embeddings. By the first (Thm. 5.6) we obtain, for each R , a semiring S which is even MC and has an identity, by the second (cf. Thm. 6.2 and Suppl. 6.3) we get semirings S which are ZDF, but neither MLC nor MRC. Moreover, these considerations and some other examples (Expl. 4.2 and 6.4, Remark 4.3) disprove various further conjectures on S , R and U and their interrelation, which have been suggested to us in the context of our investigations.

2. Preliminaries on Semirings

Let $S = (S, +, \cdot)$ be a semiring as defined in Section 1. If there exists a neutral element o of $(S, +)$ [e of (S, \cdot)], it is called the *zero* [the *identity*] of the semiring S . An element $t \in S$ is said to be *absorbing* iff $at = ta = t$ holds for all $a \in S$. It is well-known that the zero o of a semiring S need not be absorbing and may even coincide with the identity of S (cf. e.g. [20]). Conditions ensuring that the zero o of a semiring S is absorbing are that $(S, +)$ has no further idempotents or that $(S, +)$ is left or right cancellative. If $(S, +)$ has the last two properties, we call S *additively cancellative* (briefly AC).

Since we consider semirings as (2,2)-algebras $(S, +, \cdot)$, concepts as *subsemirings*, *homomorphisms* etc. are clear and refer merely to the two binary operations, also for semirings which have a zero or an identity. For subsets A, B of S , we define

$$A + B = \{a + b \mid a \in A, b \in B\} \quad \text{and} \quad AB = \{ab \mid a \in A, b \in B\}.$$

In particular, $A \neq \emptyset$ is called an *ideal* of S if $A + A \subseteq A$, $SA \subseteq A$ and $AS \subseteq A$ are satisfied (cf. e.g. [3]).

A semiring S is called *multiplicatively left cancellative* (MLC) iff all $a \in S$ or, for a semiring S with a zero o , all $a \neq o$ of S are left cancellable in (S, \cdot) . In the second case this implies (cf. [23]): either the zero o is also left cancellable in (S, \cdot) , or o is (from both sides) absorbing. Hence a non-trivial semiring S with an absorbing zero is MLC iff (S^*, \cdot) is a left cancellative subsemigroup of (S, \cdot) .

The dual concept and statements for a *multiplicatively right cancellative* (MRC) semiring S are clear, and S is called *multiplicatively cancellative* (MC) iff it is MLC and MRC. Now assume that S has a zero. Then, as already mentioned in Section 1, each of these properties implies that S is ZDF, but not conversely (cf. [5]), contrasting the situation with rings for which all four properties are equivalent.

It is well-known that an absorbing zero can be adjoined to each semiring:

Lemma 2.1. *Let $T = (T, +, \cdot)$ be any semiring and o an element not contained in T . Extend the operations on T to those on $S = T \cup \{o\}$ by*

$$a + o = o + a = a \quad \text{and} \quad a \cdot o = o \cdot a = o \quad \text{for all } a \in S.$$

Then $(S, +, \cdot)$ is a semiring with o as absorbing zero, which is ZDF, without zero sums, and contains $T = S^$ as a subsemiring. Moreover: S has commutative addition or multiplication iff this holds for T ; S is AC iff T is AC and has no zero; S is MLC iff T is MLC and has no absorbing zero or consists only of one element.*

The semiring $(\mathbb{N}, +, \cdot)$ of positive integers is in a natural way a (left and right) operator domain for each additively commutative semiring $(S, +, \cdot)$ according to

$$(2.1) \quad \nu s = s\nu = \sum_{i=1}^{\nu} s \quad \text{for all } \nu \in \mathbb{N}, s \in S.$$

The obvious rules $\nu(s+r) = \nu s + \nu r$, $(\nu + \mu)s = \nu s + \mu s$, $(\nu\mu)s = \nu(\mu s)$ and $1s = s$ show that $(S, +)$ is a unitary left (and right) \mathbb{N} -semimodule, and one also has $\nu(sr) = (\nu s)r = s(\nu r)$. If S has an absorbing zero o , the semiring $(\mathbb{N}_0, +, \cdot)$ of non-negative integers is also such an operator domain if one extends (2.1) by $0s = o$. Moreover, the ring of integers $(\mathbb{Z}, +, \cdot)$ operates a corresponding way on each ring.

Generalizing a concept introduced for rings in [6], an element a of a semiring S is called an α -*filter* of S for some $\alpha \in \mathbb{N}$ iff

$$(2.2) \quad as = sa = \alpha s \quad \text{holds for all } s \in S.$$

The original purpose of this concept was to describe the epimorphisms φ which occur in Remark 5.2 in the case of rings (cf. also [19]). In the semiring case the situation is similar, but more complicated. Here we need α -fiers in the context of Thm. 3.7.

Let S be a semiring with a zero o . An element $r \in S$ is called *additively invertible* (in S) iff $r + (-r) = (-r) + r = o$ holds for some $-r \in S$, which is then uniquely determined by r . Clearly, all elements of this kind form a subgroup $(R, +)$ of $(S, +)$ with o as neutral element.

Lemma 2.2. *Let S be a semiring with zero o and R the set of all additively invertible elements of S . Then R is an ideal of S iff o is absorbing. In this case, $(R, +, \cdot)$ is a subsemiring and additively a group, but the latter need not be commutative.*

Proof. In the trivial case $R = \{o\}$, clearly $SR = \{o\} = RS$ holds iff o is absorbing. For $R \supset \{o\}$, let R be an ideal of S and $a \in S$. Then $o + o = o$ implies $ao + ao = ao$, so that ao is an idempotent in the group $(R, +)$. This yields $ao = o$, and $oa = o$ follows in the same way. Conversely, let o be absorbing, $a \in S$ and $r \in R$. Then

$$r + (-r) = (-r) + r = o \text{ yields } ar + a(-r) = a(-r) + ar = o,$$

which proves $a(-r) = -(ar)$ and hence $ar \in R$. So we have $SR \subseteq R$ and correspondingly $RS \subseteq R$. For the last statement we note that there are various semirings $(R, +, \cdot)$ such that $(R, +)$ is a non-commutative group, also called additively not commutative rings (cf. [9] and [22], the latter also for more references). However, semirings $(R, +, \cdot)$ of this kind are never ZDF.

As a contrast to the situation in Thm. 3.3, we show that a semiring S may contain subrings R_1, R_2, \dots which have different zeroes o_1, o_2, \dots , even if S is a ZDF semiring with an absorbing zero:

Example 2.3. Consider a distributive lattice (L, \cup, \cap) as a semiring $(L, +, \cdot)$ and let $T = \{(r, l) | r \in R, l \in L\}$ be the semiring obtained as the direct product of a ring R with L . Then, for each $l_i \in L$, T contains $R_i = \{(r, l_i) | r \in R\}$ as a subring isomorphic to R , and all corresponding zeroes (o, l_i) are distinct. By Lemma 2.1 one obtains from T a semiring S as claimed above.

3. ZDF Semirings with an Absorbing Zero

In this section we investigate the structure of semirings as indicated by the title. According to the introduction we have to assume that such a semiring has zero sums, since otherwise nothing can be said beyond Lemma 2.1. The following statement will be used several times.

Lemma 3.1. *Let S be any semiring and $a, b, s, r \in S$. If*

$$(3.1) \quad as \text{ is left and } br \text{ is right cancellable in } (S, +),$$

then $ar + bs = bs + ar$ holds.

Proof. Applying the distributive laws to $(a + b)(s + r)$ in both orders of succession, we obtain

$$as + ar + bs + br = as + bs + ar + br,$$

which yields our statement by the assumptions on as and br .

Lemma 3.2. *Let S be a ZDF semiring with an absorbing zero o . Then*

$$(3.2) \quad r + r' = o \text{ implies } r' + r = o \text{ for all } r, r' \in S.$$

Proof. Since (3.2) is trivial for $r = o$, we assume $r \neq o$ and apply Lemma 3.1 for $a = b = r$ and $s = r'$. From $rr + rr' = o$ it follows that $as = rr'$ is left and $br = rr$ is right cancellable in $(S, +)$, which yields $rr + rr' = rr' + rr$. Now $r + r' = o$ implies $rr + rr' = o$. So we get $r(r' + r) = o$ for $r \neq o$, hence $r' + r = o$ as S is ZDF.

Theorem 3.3. *Let S be a ZDF semiring with an absorbing zero o . Then S has zero sums iff S contains a non-trivial subring with o as its zero. If this is the case,*

$$(3.3) \quad R = \{r \in S \mid r + r' = o \text{ or } r' + r = o \text{ for some } r' \in S\}$$

is the greatest subring of S , and even an ideal of S .

Now suppose additionally that S is a proper semiring with zero sums. Then $\{o\} \subset R \subset S$ holds for R as above and $U = S \setminus R$ satisfies

$$(3.4) \quad U + S \subseteq U, \quad S + U \subseteq U \quad \text{and hence} \quad U + U \subseteq U,$$

and each element $s \neq o$ of S has infinite additive order. Thus R and the subsemigroup $(U, +)$ of $(S, +)$ are infinite. Moreover, for any $s, t \in S$ and $\alpha \in \mathbb{N}$,

$$(3.5) \quad \alpha s + ts = o \text{ or } \alpha s + st = o \text{ imply } s = o.$$

Proof. If S contains any subring $R' \supset \{o\}$, clearly S has zero sums. Conversely, the latter implies $R \supset \{o\}$ for the set R defined by (3.3). Applying Lemma 3.2, we obtain that R consists of all additively invertible elements of S . Hence, by Lemma 2.2, R is a subsemiring and an ideal of S , and $(R, +)$ is a group. To show that $(R, +)$ is commutative, we consider the commutator $p+q+(-p)+(-q)$ for any $p, q \in R$. Again by Lemma 3.2, we have $qr + (-p)r = (-p)r + qr$ for some $r \neq o$ of R , since qr and $(-p)r$ are in R and hence cancellable in $(S, +)$. So we obtain $(p+q-p-q)r = pr + qr - pr - qr = pr - pr + qr - qr = o$ and thus $p+q-p-q = o$ since S is ZDF. So $(R, +)$ is commutative, hence $(R, +, \cdot)$ a ring and obviously the greatest subring of S which contains o . In fact, the latter restriction is superfluous since each subring of S contains o . This is clear if $R = S$ holds, which was not excluded so far, and will follow as a by-product from the following considerations for $R \neq S$.

Now we assume $U = S \setminus R \neq \emptyset$. Then $u + s \in U$ holds for all $u \in U$ and $s \in S$. Otherwise, $u + s = r \in R$ would yield $u + s + (-r) = o$ and thus the contradiction $u \in R$ by (3.3). The other statement of (3.4), $S + U \subseteq U$, follows in the same way. Next we show that each $s \neq o$ of S has infinite additive order (which also yields that any subring of S must have o as its zero). By way of contradiction, assume at first $\nu r = r + \dots + r = o$ for some $r \neq o$ of R and some $\nu \in \mathbb{N}$. Then $(\nu r)u = r(\nu u) = o$ holds for any $u \in U$. Since S is ZDF, we get $\nu u = u + \dots + u = o$ and thus the contradiction $u \in R$ by (3.3). Now assume that an element $u \in U$ has finite additive order, which only means that the set $\{\mu u \mid \mu \in \mathbb{N}\}$ is finite (and not necessarily $\nu u = o$ for some $\nu \in \mathbb{N}$). Then $\{(\mu u)r = \mu(ur) \mid \mu \in \mathbb{N}\}$ is also finite for any $r \neq o$ of R . Thus $ur \neq o$ of R would have finite additive order, which was already disproved.

For (3.5), assume by way of contradiction that e.g. $\alpha s + ts = o$ holds for some $\alpha \in \mathbb{N}$ and $s \neq o$. This yields $u(\alpha s) + uts = o$ for each $u \in U$,

hence $(\alpha u + ut)s = o$ and $\alpha u + ut = o$ as S is ZDF. But the latter implies $u \in R$ by (3.3), a contradiction.

Corollary 3.4. *Let S be a proper finite ZDF semiring with an absorbing zero. Then S has no zero sums.*

Supplement 3.5. *Let S be a proper ZDF semiring with an absorbing zero o and zero sums, R its greatest subring and $U = S \setminus R$.*

a) *Assume $sa = sb$ or $as = bs$ for some $s \neq o$ and $a \neq b$ of S . Then $ra = rb$ and $ar = br$ hold for each $r \in R$, and a and b are in U .*

b) *For all $r \neq o$ of R we have $rR \cap rU = \emptyset$ and hence $rR \subset R$, and correspondingly for Rr and Ur .*

c) *Assume $s + a = s + b$ or $a + s = b + s$ for some s and $a \neq b$ of S . Then s, a and b are in U and $ra = rb$ and $ar = br$ hold for each $r \in R$.*

d) *One has $U + R = R + U = U$ where $u_1 + r = u_2 + r$ implies $u_1 = u_2$ and $u + r_1 = u + r_2$ implies $r_1 = r_2$, and correspondingly for $R + U$. Moreover, there is a subset $W \subset U$ such that each $u \in U$ has a unique presentation $w + r$ for some $w \in W$ and $r \in R$.*

Proof. a) From $sa = sb$ we obtain $sar = sbr$ or $s(ar + b(-r)) = o$ for each $r \in R$, which yields $ar = br$ as S is ZDF. The latter implies $ra = rb$ for each $r \in R$ in same way. Since a ZDF ring is also MC, at most one of a and b can be in R . We may assume $a = q \in R$ and $b \in U$. But then $rq = rb$ for some $r \neq o$ would imply $r(b - q) = o$, hence the contradiction $b = q$.

b) We have just proved that $rq = rb$ for $r \neq o$ of R and $q \in R, b \in U$ is impossible, which yields $rR \cap rU = \emptyset$. Note that $rR \subset R$ is also a consequence of (3.5).

c) From $s + a = s + b$ and $a \neq b$ it follows that $s \in U$ since each element of R is clearly cancellable in $(S, +)$. For the same reason, $rs + ra = rs + rb$ implies $ra = rb$ for each $r \in R$ by $rs \in R$. The rest follows from a).

d) The first part is a consequence of (3.4), $o \in R$ and c). Next we state that

$$(3.6) \quad a + p = b + q \quad \text{for some } p, q \in R,$$

i.e. $a = b + r$ for some $r \in R$, defines clearly an equivalence $a \sim b$ on S for which R is one equivalence class. Each set W of representatives for all other classes obviously satisfies the last statement.

It is well known that, for each ideal R of a semiring S , (3.6) defines a congruence κ on $(S, +, \cdot)$ provided that $(S, +)$ is commutative (cf. [3], but observe [4]). The latter can be replaced by $u + R = R + u$ for all $u \in U = S \setminus R$. The converse question to construct all semirings S which contain R as an ideal such that the congruence class semiring S/κ (mostly denoted by S/R as in the ring case) is isomorphic to a given semiring has been settled by Rédei in [14]. The restriction to AC semirings in [14] is unessential. So we can state:

Supplement 3.6. *For S, R and U as in Suppl. 3.5, assume $u + R = R + u$ for all $u \in U$ and define κ by (3.6). Then S is an Everett-Rédei semiring extension of the ring R by the congruence class semiring $S/\kappa = S/R$, a semiring with an absorbing zero, but without zero sums, which is ZDF iff U is a subsemiring of S .*

Although those extensions are hard to handle in general, we have used the theory given in [14] as a guide-line to obtain some of our examples, which, except Expl. 6.4, are all special cases of extensions according to Suppl. 3.6.

The main purpose of these examples given in Section 5 and 6 is to prove that our statements on S, R and U in Thm. 3.3 and Suppl. 3.5 are fairly complete concerning the general situation (but cf. Thm. 4.1). In particular, we shall see that the subsemigroup $(U, +)$ need not be a subsemiring of S (cf. Remark 4.3), that only the elements of R have to commute in $(S, +)$ (cf. Thm. 6.2 and Expl. 6.4), and that all violations of cancellativity left over by a), c) and d) of our supplement really may occur (cf. Thm. 6.2). Also our statements on R are complete according to the following:

Theorem 3.7. *Let R' be a non-trivial ring with zero o . Then R' is a subring of a proper ZDF semiring S such that o is the absorbing zero of S iff R' is ZDF and satisfies the condition*

C) $\alpha s + ts = o$ implies $s = o$ for all $s, t \in R'$ and $\alpha \in \mathbb{N}$.

The condition C) can also be formulated with $\alpha s + st = o$ and is equi-

valent to the fact that R' contains no α -fier for any $\alpha \in \mathbb{N}$. It implies that each $r \neq o$ of R' has infinite additive order.

Moreover, for each ring R' of this kind, S can be chosen in such a way that R' is its greatest subring.

Proof. If R' is contained in a semiring S as assumed above, S has zero sums. Hence the greatest subring R of S is clearly ZDF and satisfies C) and thus the same holds for $R' \subseteq R$.

The converse statement including the last one will be shown in two versions, namely in Thm. 5.6 (where S is even MC) and in Thm. 6.2 (where S is merely ZDF).

Concerning the remarks on C), we consider any ZDF ring R' and assume $\alpha s + ts = o$ for any $s \neq o$. This yields $\alpha r + rt = o$ for each $r \in R$, in particular $\alpha s + st = o$, and in turn $\alpha r + tr = o$. Hence $(-t)$ is an α -fier of R' as defined by (2.2), which conversely implies $\alpha s + ts = o$ even for each $s \in R'$. The last remark is clear, since an element $s \neq o$ of R' of finite additive order satisfies $\alpha s = o$ for some $\alpha \in \mathbb{N}$, which contradicts C).

4. Multiplicative Cancellativity

Let S be a proper ZDF semiring with an absorbing zero o and zero sums. Then, according to Suppl. 3.5 a), left and right cancellativity in (S, \cdot) are closely connected. In particular, it is near by hand to ask whether there are semirings S as above which are not MLC or MRC. We have claimed that without proof in [12], Section 6, and we will show this by the following concrete Expl. 4.2 which can be checked directly (regardless that our constructions in Section 6 will provide lots of those examples as already indicated in the proof of Thm. 3.7). Before that we sharpen the situation by the following result:

Theorem 4.1. *Let S be semiring which has zero sums. Then S is MLC iff S is MRC, hence in turn iff S is MC.*

If this is the case, the zero o of S is absorbing and S is AC and additively

commutative.

Proof. Since all statements are true if S is a ring, we consider a proper semiring. We show at first that MLC implies MRC and the statement on o . So let S be MLC. Then, by a result of [23] cited in Section 2, its zero o is either also multiplicatively left cancellable in (S, \cdot) or absorbing. Assuming the former, we get from $o(a + a) = (o + o)a = oa$ that $a + a = a$ holds for each $a \in S$. But then $r + r' = o$ for any $r, r' \in S$ implies

$$r = r + o = r + r + r' = r + r' = o$$

and hence $r' = o$. This contradicts that S is assumed to have zero sums. So the zero o of S is absorbing and, since MLC yields ZDF, we can apply our results of Section 3. By way of contradiction, assume that S is not MRC. Then there are elements $s \neq o$ and $a \neq b$ of S satisfying $as = bs$. But this yields $ra = rb$ for all $r \in R \neq \{o\}$ by Suppl. 3.5 a), contradicting that S is MLC. Clearly, MRC implies MLC in the same way.

Now we assume that S is MC and that $s + a = s + b$ or $a + s = b + s$ hold for some $s \in S$ and $a \neq b$ of S . Then we obtain, by Suppl. 3.5 c), $ra = rb$ for all $r \in R \neq \{o\}$. This contradicts that S is MC and proves S to be AC. Hence Lemma 3.1 implies $ac + bc = bc + ac$ for all $a, b, c \in S$ which yields $a + b = b + a$ since S is MC.

Example 4.2. Let x^0, x^1, x^2, \dots be the elements of the free monoid (X, \cdot) generated by x with x^0 as its identity, and (H, \cdot) the semigroup obtained from (X, \cdot) by adjoining a new identity $e \notin X$. Let

$$D = \left\{ \sum_{i=0}^n \gamma_i x^i + \gamma e \mid \gamma_i, \gamma \in \mathbb{Z} \right\}$$

be the semigroup ring of (H, \cdot) over the ring \mathbb{Z} of integers. (In other words, D is obtained from the polynomial ring $\mathbb{Z}[x]$ by adjoining a new identity e , or D is the Dorroh-ring $Do(\mathbb{Z}, \mathbb{Z}[x])$ in the sense of Section 5). Clearly, D is commutative. Now

$$S = \left\{ \sum_{i=0}^n \gamma_i x^i + \gamma e \mid \gamma_0, \gamma \in \mathbb{N}_0, \gamma_i \in \mathbb{Z} \text{ for } i \geq 1 \right\}$$

is a subsemiring of D with $o \in D$ as its absorbing zero. Further, S has zero sums and its greatest subring R consists of all polynomials of $\mathbb{Z}[x]$ satisfying $\gamma_0 = 0$. Moreover, $(1x)(1x^0) = (1x)(1e)$ shows that S is not MC. So it remains to prove that S is ZDF, which is easily checked in a straightforward way.

Remark 4.3. In Expl. 4.2, $U = S \setminus R$ consists of all elements of S satisfying $\gamma_0 + \gamma \neq 0$, and U is a subsemiring of S . But we can change the definition of S e.g. by $\gamma_1, \gamma_0, \gamma \in \mathbb{N}_0$, but also by $\gamma_1, \gamma_0 \in \mathbb{N}_0$ and $\gamma = 0$ (and, clearly, $\gamma_i \in \mathbb{Z}$ for $i \geq 2$). In both cases S remains a ZDF semiring with zero sums, where the greatest subring R consists now of all polynomials of $\mathbb{Z}[x]$ satisfying $\gamma_0 = \gamma_1 = 0$. Hence $U = S \setminus R$ contains in both cases the element $1x$, and $(1x)(1x) = 1x^2 \in R$ shows that U is not a subsemiring of S . Note that S is MC in this second variation of Expl. 4.2, but not in the first one.

Together with statements of Section 3, we obtain a further result from Thm. 4.1. Recall for this purpose that a semiring S is embeddable into a ring iff S is AC and additively commutative (where the former yields that the zero of S is absorbing, if there is one). If this is the case, there exists, unique up to isomorphisms, a smallest ring which contains S as a subsemiring. This ring is called the *difference ring* of S and denoted by $D(S)$, since it consists of all differences $a - b$ for $a, b \in S$, subject to elementary rules.

Now let S be a semiring such that $D(S)$ exists. If S is ZDF but not MRC or even MLC but not MRC, then clearly the properties ZDF or MLC of S do not transfer to $D(S)$. (Otherwise, since ZDF, MLC and MRC are equivalent for the ring $D(S)$, such a transfer would yield that S is also MRC. Cf. also Expl. 4.2 in this context.) But even if S has all these properties, i.e. if S is MC, its difference ring $D(S)$ need not be MC. E.g., consider the congruence class ring $T = \mathbb{Z}[x]/(x^2)$ of the polynomial ring $\mathbb{Z}[x]$ and let S consist of all classes which can be represented by some $\gamma_0 + \gamma_1 x$ for $\gamma_0 > 0$ and $\gamma_1 \geq 0$. Then one checks that S is a MC subsemiring of T , whereas $D(S) = T$ is clearly not MC (cf. [21], p. 221).

Theorem 4.4. *Let S be a semiring with zero sums which is MLC (or MRC). Then S is embeddable into a ring, and the smallest ring $D(S)$*

containing S is MC.

Proof. If S itself is a ring, there is nothing to prove. If S is a proper semiring, we apply Thm. 4.1. Hence S is AC and additively commutative, and so a subsemiring of its difference ring $D(S)$. It remains to show that the latter is ZDF and thus MC (which, according to the above counter-example, depends on some further assumption on S , in our case the existence of zero sums). By way of contradiction, assume $(a - b)(c - d) = o$ for some $a - b \neq o$ and $c - d \neq o$ of $D(S)$. Note that S satisfies all conditions such that it has a greatest subring $R \neq \{o\}$ according to Thm. 3.3. So we obtain $(ra - rb)(cr - dr) = o$ for some $r \neq o$ of R , where ra, rb, cr and dr are in R and hence also $ra - rb$ and $cr - dr$. Since R is MC we get that e.g. $ra - rb = o$ holds, which yields $a = b$ in S , hence the contradiction $a - b = o$.

5. Embedding into Semirings with an Identity

Considerations according to the title will also lead to all our constructions of ZDF semirings with zero sums. For this purpose we need explicitly the well-known result due to Dorroh (cf. [8]) that each ring R can be embedded into a ring with identity in the following way. One defines operations on the set $D = \mathbb{Z} \times R$ by

$$(5.1) \quad (\nu, s) + (\mu, t) = (\nu + \mu, s + t) \quad \text{and}$$

$$(5.2) \quad (\nu, s) \cdot (\mu, t) = (\nu \cdot \mu, \nu t + \mu s + s \cdot t),$$

where νt and μs are defined according to (2.1). Then $(D, +, \cdot)$ is a ring with $(1, o) = e$ as identity. By an obvious isomorphism, one can identify $(0, s)$ with s for each $s \in R$ so that R becomes a subring of D , which also yields the unique presentation

$$(5.3) \quad (\nu, s) = \nu(1, o) + (0, s) = \nu e + s \quad \text{for the elements of } D.$$

We call this ring $D = \mathbb{Z}e + R$ the *Dorroh-ring* of R and denote it by $Do(\mathbb{Z}, R)$. It is universal in the sense that each ring $\mathbb{Z}e' + R$ generated by R and an identity e' is an R -epimorphic image of $Do(\mathbb{Z}, R)$ (cf. [6],[19], and the corresponding Remark 5.2 for semirings).

It is also known that a semiring with non-commutative addition need not be embeddable into one with an identity (cf. [10], but observe [11]), whereas the above statements can be transferred to additively commutative semirings (cf. e.g. [16]):

Proposition 5.1. *Let S be a semiring with an absorbing zero o and commutative addition. Then the above construction applied to $D = \mathbb{N}_o \times S$ yields a proper additively commutative semiring $(D, +, \cdot)$ with $(0, o)$ as absorbing zero and $(1, o) = e$ as identity. Obviously, the subsemiring $\{(0, s) | s \in S\}$ of D is isomorphic to S and can be replaced by the latter, which yields $D = \mathbb{N}_o e + S$ according to (5.3).*

We call this semiring the *Dorroh-semiring* of S and denote it by $Do(\mathbb{N}_o, S)$. Observe also that $Do(\mathbb{N}_o, S)$ is AC iff S is AC.

(Clearly, Prop. 5.1 applies also to an arbitrary additively commutative semiring T via Lemma 2.1).

Remark 5.2. The semiring $D = Do(\mathbb{N}_o, S)$ is universal in the following sense. Let S be a subsemiring of any additively commutative semiring \bar{T} with an identity, say e' . Then

$$T = \mathbb{N}_o e' + S = \{\nu e' + s | \nu \in \mathbb{N}_o, s \in S\}$$

is a subsemiring of \bar{T} with $o \in S$ as absorbing zero and e' as identity, and there is an epimorphism

$$\varphi : (D, +, \cdot) \rightarrow (T, +, \cdot) \text{ given by } \nu e + s \rightarrow \nu e' + s.$$

Since φ leaves each $s \in S$ fixed, we call it an *S-epimorphism*. One checks that a typical example of such an epimorphism φ satisfying $\varphi(\alpha e') = \varphi(a)$ for a fixed α -fier a of S according to (2.2) is obtained from the congruence on D defined by

$$(\gamma - \sigma\alpha, \sigma\alpha + c) \equiv (\gamma - \tau\alpha, \tau\alpha + c)$$

for any $(\gamma, c) \in D$ and any $\sigma, \tau \in \mathbb{N}_o$ satisfying $\gamma \geq \tau\alpha$ and $\gamma \geq \sigma\alpha$.

Now we obtain a rather general result on ZDF semirings:

Theorem 5.3. *Each proper ZDF semiring S with commutative addition and an absorbing zero o can be embedded into a semiring of the*

same kind which has an identity. In particular, the Dorroh-semiring $Do(\mathbb{N}_0, S)$ of S is such a semiring.

Proof. We only have to show that $Do(\mathbb{N}_0, S)$ is ZDF. By way of contradiction, assume $(\nu, s)(\mu, t) = (0, o)$ for some $(\nu, s) \neq (0, o) \neq (\mu, t)$ of $Do(\mathbb{N}_0, S)$. To obtain $\nu\mu = 0$ in (5.2), we assume at first $\nu = 0$, which yields $s \neq o$ and $\mu s + st = o$. Clearly, $\mu = 0$ and hence $t \neq o$ contradicts that S is ZDF. But $\mu \neq 0$ implies that the proper semiring S has zero sums.

So we can apply Thm. 3.3, where (3.5) states that $\mu s + st = o$ for $\mu \in \mathbb{N}$ yields $s = o$, again a contradiction. The case $\mu = 0$ follows in the same way via $\nu t + st = o$.

Remark 5.4. Due to [16], the first part of Thm. 5.3 remains true if one replaces ZDF by MC. However, $D = Do(\mathbb{N}_0, S)$ itself need not be MC if S is. In the contrary, there is a unique S -epimorphic image $T = D/\kappa$ of D which is MC (cf. Remark 5.2), where the corresponding congruence κ on D is given by

$$(\nu, s)\kappa(\nu', s') \text{ iff } \nu t + ts = \nu' t + ts' \text{ for some } t \neq o \text{ of } S.$$

Remark 5.5. The first part of Thm. 5.3 remains also true if $S = R$ is a ZDF ring and hence also MC. But again the semiring $Do(\mathbb{N}_0, R)$ as well as the ring $Do(\mathbb{Z}, R)$ need not be ZDF. Due to [17], there is a unique R -epimorphic image $\mathbb{Z}e' + R \cong Do(\mathbb{Z}, R)/\mathfrak{a}$ of the Dorroh-ring by a suitable ideal \mathfrak{a} , the smallest ZDF ring containing R and an identity (cf. also [19]). Clearly, the R -epimorphism of $Do(\mathbb{Z}, R)$ induces one for its subsemiring $Do(\mathbb{N}_0, R)$.

A special case of the last remark provides, as announced in Section 3, our first construction of a ZDF semiring which contains a given ring R (satisfying the necessary conditions) as its greatest subring:

Theorem 5.6. *Let R be a non-trivial ZDF ring which satisfies the condition C) of Thm. 3.7. Then the Dorroh-semiring $Do(\mathbb{N}_0, R)$ of R is a proper ZDF semiring S with zero sums containing R as its greatest subring. In fact, S is even MC and contains an identity.*

Proof. Clearly, R is the greatest subring of $Do(\mathbb{N}_0, R)$, and it remains to show that $Do(\mathbb{N}_0, R)$ is MC, due to the conditions on R . For the

latter, we prove that the Dorroh-ring $Do(\mathbb{Z}, R)$ is ZDF and hence MC. By way of contradiction, assume $(\nu, s)(\mu, t) = (0, o)$ for some $(\nu, s) \neq (0, o) \neq (\mu, t)$ of $Do(\mathbb{Z}, R)$. There is no loss of generality in assuming that ν and μ are in \mathbb{N}_0 . So we can use the proof of Thm. 5.3 and obtain for $\nu = 0$ clearly $s \neq o$, but also $\mu s + st = o$ for some $\mu \in \mathbb{N}$, which contradicts C). For $\mu = 0$ and $t \neq o$ we get $\nu t + st = o$ for some $\nu \in \mathbb{N}$, which is also excluded by C) and completes our proof.

Note that each MC semiring S which contains R as a subring such that both have the same absorbing zero has to be AC and additively commutative by Thm. 4.1. So the fact that $S = Do(\mathbb{N}_0, R)$ has these properties corresponds to this situation.

6. Further Constructions of ZDF Semirings

Our next point is to show that each ring R satisfying the necessary conditions of Thm. 3.7 is the greatest subring of a proper ZDF semiring which is not MLC (and hence not MRC by Thm. 4.1). For this purpose we generalize the construction of the Dorroh-semiring $Do(\mathbb{N}_0, S)$ given in Prop. 5.1 as follows.

Construction 6.1. Let W_0 be any semiring with an absorbing zero o and ψ a homomorphism of $(W_0, +, \cdot)$ into $(\mathbb{N}_0, +, \cdot)$. To simplify our notation, we write $\psi(v) = |v|$ for each $v \in W_0$. Note that $|o| + |o| = |o|$ yields $|o| = 0$. Let S be a semiring with commutative addition and an absorbing zero, also denoted by o . Then we define operations on the set $D = W_0 \times S$ in replacing (5.1) and (5.2) by

$$(6.1) \quad (v, s) + (w, t) = (v + w, s + t) \quad \text{and}$$

$$(6.2) \quad (v, s) \cdot (w, t) = (v \cdot w, |v|t + |w|s + s \cdot t),$$

where $|v|t$ and $|w|s$ are defined by the natural operation (2.1) of \mathbb{N}_0 on S . It is straightforward to check that $(D, +, \cdot)$ is a semiring with (o, o) as absorbing zero. Moreover, by obvious isomorphisms, we can identify (o, s) with s for each $s \in S$ and (v, o) with v for each $v \in W_0$. Then W_0 and S become subsemirings of D , and we have the unique

presentation

$$(6.3) \quad (v, s) = (v, o) + (o, s) = v + s \quad \text{for the elements of } D.$$

We denote this semiring by $Do(W_0, \psi, S)$.

Theorem 6.2. *Let R be a non-trivial ZDF ring which satisfies the condition C) of Thm. 3.7. Let W_0 be a non-trivial semiring with a zero o and $\psi : W_0 \rightarrow \mathbb{N}_0$ a homomorphism satisfying*

$$(6.4) \quad \psi(v) = |v| = 0 \iff v = o \quad \text{for all } v \in W_0,$$

which yields that the zero o of W_0 is absorbing and that W_0 is ZDF and has no zero sums.

Then the semiring $S = Do(W_0, \psi, R)$ constructed above with $(o, o) = o$ as absorbing zero is ZDF and has zero sums, and $R = \{(o, r) | r \in R\}$ is its greatest subring. Clearly, S is additively commutative or AC iff W_0 has the same property, and S has an identity, namely (e, o) , iff W_0 has an identity e . However, S is neither MLC nor MRC iff there are elements $w \neq w'$ of W_0 satisfying $|w| = |w'|$.

Proof. All statements on W_0 claimed as consequences of (6.4) are checked straightforwardly. In particular, W_0 has no zero sums. Hence, by (6.1), all zero sums of S are of the form $(o, r) + (o, -r) = (o, o)$ for some $r \in R$, hence R is the greatest subring of S . Next we show that S is ZDF and assume, by way of contradiction, $(v, s)(w, t) = (o, o)$ for some $(v, s) \neq (o, o) \neq (w, t)$. From $vw = o$ by (6.2) and since W_0 is ZDF, we get $v = o$ or $w = o$, and it is enough to consider the first case. Then we get $s \neq o$ and, by (6.2), $|w|s + st = o$, which yields $|w| = 0$ due to the assumed condition C) for R . But the latter implies $w = o$ by (6.4), and $st = o$ yields $t = o$ since R is ZDF. Thus we have the contradiction $(w, t) = (o, o)$. Finally, if ψ is injective, $Do(W_0, \psi, R)$ is R -isomorphic to a subsemiring of $Do(\mathbb{N}_0, R)$ and hence MC by Thm. 5.6. Otherwise, there are $w \neq w'$ in W_0 satisfying $|w| = |w'|$, which yields $(o, s)(w, t) = (o, s)(w', t)$ for all $s \neq o$ of R . So S is not MLC and hence not MRC by Thm. 4.1.

We remark without proof that all elements $(v, s) \in S$ with $v \neq o$ are multiplicatively left as well as right cancellable in S .

Note that each element of the semiring $S = Do(W_0, \psi, R) = W_0 + R$ has the unique presentation $(v, r) = v + r$ for $v \in W_0$ and $r \in R$, according to (6.3). In particular, $W = W_0 \setminus \{o\}$ is a subset of $U = S \setminus R$ such that $u = w + r$ is the unique presentation of the elements of U as described in Suppl. 3.5 d). So, in order to obtain by Thm. 6.2 embeddings of R into ZDF and not MC semirings S , as we have announced in Section 3, it remains to show:

Supplement 6.3. *There are semirings W_0 with a zero o which have non-injective homomorphisms $\psi : W_0 \rightarrow \mathbb{N}_0$ satisfying (6.4). In particular, there are semirings W_0 of this kind which are not AC or not additively commutative.*

Proof. Let A be any non-trivial semiring and W the direct product of \mathbb{N} and A , which means that the set $W = \{(\nu, a) | \nu \in \mathbb{N}, a \in A\}$ is endowed with componentwise addition and multiplication. Let W_0 be obtained from W by adjoining an absorbing zero $o \notin W$ according to Lemma 2.1. Then $\psi((\nu, a)) = \nu$ for all $(\nu, a) \in W$ and $\psi(o) = 0$ define obviously a homomorphism $\psi : W_0 \rightarrow \mathbb{N}_0$ as claimed above. Moreover, W_0 is AC or additively commutative iff A has the same property. There are clearly semirings which even violate both properties. E.g., let A be any set of at least two elements and define $a + b = a$ and $a \cdot b = c$ for all $a, b \in A$ and any fixed element $c \in A$.

As noted above, for each semiring $S = Do(W_0, \psi, R)$ obtained in this way by Thm. 6.2 and Suppl. 6.3, W is a subsemiring of $U = S \setminus R$. Hence we also see by the last statements that $(U, +)$ need neither be cancellative nor commutative (observe again Thm. 4.1 in this context). So we have settled all our announcements given before Thm. 3.7 concerning the structure of proper ZDF semirings S satisfying $S \supset R \supset \{o\}$ as considered in Thm. 3.3 and Suppl. 3.5, except an example such that $u + r \neq r + u$ holds for some $u \in U$ and $r \in R$. Although this could also be done in a rather general way, we restrict ourselves to present a concrete case.

Example 6.4. On $W_0 = 2\mathbb{N}_0 \times 2\mathbb{N}_0$, where $2\mathbb{N}_0$ denotes the set of even non-negative integers, define operations by

$$(6.5) \quad (\nu_1, \nu_2) + (\mu_1, \mu_2) = (\nu_1 + \mu_1, \nu_2 + \mu_2) \quad \text{and}$$

$$(6.6) \quad (\nu_1, \nu_2) \cdot (\mu_1, \mu_2) = (0, (\nu_1 + \nu_2)(\mu_1 + \mu_2)).$$

One easily checks that $(W_0, +, \cdot)$ is a semiring with $(0, 0) = o$ as absorbing zero, and that $\psi(\nu_1, \nu_2) = \nu_1 + \nu_2$ defines a homomorphism $\psi : W_0 \rightarrow \mathbb{N}_0$ which is not injective and satisfies (6.4). Note that there is an automorphism χ of W_0 given by $\chi(\nu_1, \nu_2) = (\nu_2, \nu_1)$ which satisfies

$$(6.7) \quad \psi(\chi(v)) = \psi(v) \text{ for all } v = (\nu_1, \nu_2) \in W_0.$$

Let R be the subring of $\mathbb{Z}[x]$ given by

$$R = \left\{ f(x) = \sum_{i=1}^n \gamma_i x^i \mid \gamma_i \in \mathbb{Z} \right\}.$$

Then $Do(W_0, \psi, R) = \{(v, f(x)) \mid v \in W_0, f(x) \in R\}$ provides a ZDF semiring $(S, +, \cdot)$ according to Thm. 6.2 which contains R as its greatest subring, whose addition is of course commutative. The semiring we want to construct will be (S, \oplus, \cdot) , obtained from $(S, +, \cdot)$ in defining a new addition by

$$(v, f(x)) \oplus (w, g(x)) = \begin{cases} (v + w, f(x) + g(x)) & \text{if } \gamma_1 \text{ is even} \\ (v + \chi(w), f(x) + g(x)) & \text{if } \gamma_1 \text{ is odd,} \end{cases}$$

where $\gamma_1 \in \mathbb{Z}$ denotes the coefficient of x^1 in $f(x)$. This clearly yields $f(x) \oplus w = \chi(w) \oplus f(x) \neq w \oplus f(x)$ for all $w \neq o$ of $W \subseteq U = S \setminus R$ and all $f(x) \in R$ for which γ_1 is odd.

To show that (S, \oplus, \cdot) is again a semiring, one has to check that (S, \oplus) is a semigroup and, since (S, \cdot) is commutative, one of the two distributive laws. This can be done in a straightforward manner. But we note that the associativity of (S, \oplus) is known, since the latter is a special semidirect product of the semigroups $(W_0, +)$ and $(R, +)$. Moreover, the distributivity depends essentially on (6.7) and on the restriction of W_0 to pairs of even integers, the latter since then in (6.2),

$$(v, f(x))(w, g(x)) = (v \cdot w, |v|g(x) + |w|f(x) + f(x) \cdot g(x)),$$

the crucial coefficient of x^1 of the polynomial of the right hand side is always even.

Clearly, (S, \oplus, \cdot) is a ZDF semiring like $(S, +, \cdot)$, and $(R, \oplus, \cdot) = (R, +, \cdot)$ is again the greatest subring of (S, \oplus, \cdot) . But $w \in U \setminus S$ and $f(x) \in R$ do not always commute as noted above.

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