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AN EMBEDDING THEOREM FOR FREE ASSOCIATIVE ALGEBRAS

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For Hiroyuki Tachikava on his 60th birthday.

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Abstract: Embeddings of a free algebras of countable rank in free algebras of rank two are studied, which possess special properties, such as inertia or honesty. Their existence has the consequence that the embedding can be extended to one of their universal fields of fractions.

1. Introduction

It is well known, and easily verified, that a free (associative) algebra F of countable rank can be embedded in a free algebra G of rank 2; thus in $k < x, y >$ the elements $y^n x$ $(n = 0, 1, \dots)$ freely generate a free subalgebra. But frequently one needs embeddings with special properties, and here the above example is usually insufficient. Thus one may want embeddings $F \to G$ with one or more of the following properties:

1. 1-inert embeddings. This means that if $c \in F$ has a factorization $c = ab$ in G, then there exists a unit u in G such that $au, u^{-1}b \in F$ (for simplicity we have here identified F with its image in G.)

2. More generally, an n-inert embedding is the case where the matrix ring $\mathcal{M}_n(F)$ is 1-inertly embedded in $\mathcal{M}_n(G)$.

3. Honest embeddings. Their definition will be given below in §2; it amounts to requiring the universal field of fractions of F to be embedded in that of G.

The existence of 1-inert embeddings of F in G was conjectured by G.M. Bergmann in [1] and later proved (though not published) by him (cf. Th. 4.5.3, p. 217 of [5]). Our aim in this note is to construct an honest embedding of F in G, in §2, and to give an illustration, in §3. Our construction also provides an example of a 1-inert embedding which is simpler than that in [5]; whether it is n-inert for $n > 1$ remains open.

2. An honest embedding

Throughout, all rings are associative with 1, which is preserved by homomorphisms and inherited by subrings. Fields need not be comutative; sometimes the prefix 'skew' is used for emphasis.

A matrix C over a ring R is said to be *full* if it is square, say $n \times n$ and cannot be written in the form $C = PQ$, where P is $n \times r$, Q is $r \times n$ and $r < n$. A homomorphism of rings is called *honest* if it keeps full matrices full. Since every non-zero element, as 1×1 matrix, is full, an honest homomorphism is necessarily injective. To give an example, in the embedding $F \to G$ described in §1, with $z_n \mapsto y^n x$ say,

$$\begin{pmatrix} z_0 & z_1 \\ z_2 & z_3 \end{pmatrix} \text{ maps to } \begin{pmatrix} x & yx \\ y^2x & y^3x \end{pmatrix} = \begin{pmatrix} 1 \\ y^2 \end{pmatrix} (x \ yx),$$

and this shows that the mapping is not honest.

The importance of full matrices is that certain classes of rings such as semifirs have the property that each can be embedded in a skew field over which every full matrix from the ring becomes invertible. This is called the *universal field of fractions* of the ring and for a ring R it will be denoted by $U(R)$ (cf. [5], Ch. 7 for details). Thus if we have a homomorphism of semifirs $\beta : F \to G$, we can form a commutative square as shown precisely when β is honest, and one method of establishing that β is honest is to prove the existence of such a commutative square.

$$\begin{array}{ccc} \mathrm{F} & \xrightarrow{\ \beta\ } & \mathrm{G} \\ \downarrow & & \downarrow \\ U(F) & \xrightarrow[\ \beta^*\ ]{} & U(G) \end{array}$$

Let D be a skew field and K a subfield. By the *tensor-D-ring* over K on a set X we understand the ring generated by D (as ring) and X, with the defining relations

$$\alpha x = x\alpha \quad \text{for all} \quad x \in X,\ \alpha \in K. \tag{1}$$

This ring will be denoted by $D_K < X >$; when $D = K$, it reduces to the free K-ring $K < X >$ (called a *free K-algebra* when K is commutative), and the free D-ring over K may be expressed as a free product (coproduct)

$$D_K < X >= D_K^* \ K< X > .$$

Frequently it is assumed that K is contained in the centre of D, but we shall not make this assumption here. We must then bear in mind that we cannot substitute arbitrary elements of D for X (because the relations (1) might then be violated), but we have to restrict X to values in the centralizer of K.

We shall write $F = K < Z >$, where $Z = \{z_0, z_1, \ldots\}$, $G = k < x, y >$; as indicated above, to find an honest embedding of F in G we only need an embedding of $U(F)$ in $U(G)$. Such an embedding was described in [2] (cf. [3], p. 120), but under that mapping the image of F was not confined to G. It was obtained by finding an automorphism permuting Z transitively, and realizing this automorphism by conjugation in G. We shall find that the same purpose can be achieved by a derivation, and this time the image of F stays within G. We first describe the derivation.

Lemma 2.1. *The free K-ring $K < Z >$, where $Z = \{z_0, z_1, \ldots\}$, has a derivation δ over K such that $z_i^\delta = z_{i+1}$, and δ extends to a unique derivation of $U(F)$.*

Proof. The mapping

$$\delta : z_i \mapsto z_{i+1} \quad (i = 0, 1, \ldots) \tag{2}$$

extends to a unique derivation of F because F is free (Prop. 1 of 3.3, p. 67 of [4]). Thus we have a derivation δ of F satisfying (2). We can write this as a homomorphism form F to $M_2(F)$:

$$\Delta : a \mapsto \begin{pmatrix} a & a^\delta \\ 0 & a \end{pmatrix}; \tag{3}$$

it induces a homomorphism from $\mathcal{M}_n(F)$ to $\mathcal{M}_{2n}(F)$ such that every full matrix over F maps to an invertible matrix over $U(F)$. For if A is full over F, then it is invertible over $U(F)$, hence

$$\begin{pmatrix} A & A^\delta \\ 0 & A \end{pmatrix} \quad \text{has the invers} \quad \begin{pmatrix} A^{-1} & -A^{-1}A^\delta A^{-1} \\ 0 & A^{-1} \end{pmatrix}.$$

Therefore the homomorphism Δ can be extended to a unique homomorphism from $U(F)$ to $\mathcal{M}_2(U(F))$, again denoted by Δ. Clearly it has again the form (3) and the (1,2)-entry is a derivation of $U(F)$ extending δ; it is unique because the extension of Δ was unique.
We can now obtain the desired embedding by realizing δ as an inner derivation. As usual we write $[a, b] = ab - ba$.

Theorem 2.2. *Let $G = K < x, y >$, $F = K < Z >$, where $Z = \{z_0, z_1, \ldots\}$ and K is a skew field. Then there is an embedding $\beta_0 : F \mapsto G$ defined by*

$$z_0 \mapsto y, \; z_1 \mapsto [y, x], \; z_2 \mapsto [[y, x], x], \; \ldots \tag{4}$$

If the image of F is denoted by F_0, then $G = \oplus_{n=0}^{\infty} F_0 x^n$. Moreover, the embedding is 1-inert and honest.

Proof. Let δ be the derivation of F defined as in Lemma 2.1, (1), and denote by H the skew polynomial ring $H = F[x; 1, \delta]$ with the commutation rule

$$ax = xa + a^\delta \quad \text{for all} \quad a \in F. \tag{5}$$

Then we have by (2) and (5), $z_{i+1} = z_i^\delta = z_i x - x z_i = [z_i, x]$. We claim that H is the free K-ring on x, z_0. For it is clearly generated by x and z_0 over K; to show that x, z_0 are free generators, we establish a homomorphism $\beta : H \to G$ such that $x \mapsto x$, $z_0 \mapsto y$. We begin by defining $\beta : Z \to G$ by

$$\beta : z_n \mapsto [\dots[y,x],\dots,x] \text{ with } n \text{ factors } x.$$

Since F is free on Z, this mapping extends to a homomorphism $\beta_0 : F \to G$. Moreover, we have $z_n^\delta \beta_0 = z_{n+1}\beta_0 = [\dots[y,x],\dots,x] = [z_n\beta_0, x]$ (where there are $n+1$ factors x). Hence if δ_x is the inner derivation defined by x in G, we have $\delta\beta_0 = \beta_0\delta_x$. Now the defining relations of H in terms of F are just the equations (5), which may be written $\delta = \delta_x$. Hence on H we have $\delta_x\beta_0 = \beta_0\delta_x$; thus the defining relations of H are preserved by β_0 and so β_0 may be extended to a homomorphism β of H into G. Since G is free on x, y, this shows H to be free on x, z_0, as claimed. Moreover, we see that β is surjective, hence it is an isomorphism between H and G, and the structure of H as skew polynomial ring over F shows that $G = \oplus_{n=0}^{\infty} F_0 x^n$, where F_0 is the image of F under β_0.

We now repeat the construction with $U(F)$ instead of F, thus we form the ring $U(F)[x;1,\delta]$; this is justified by the fact that δ is defined on $U(F)$. This gives us a skew polynomial ring over a field, and we can form its field of fractions $L = U(F)\ (x;1,\delta)$. Since $H = F[x;1,\delta]$ is generated over K by x, z_0, it follows that L is generated by x, z_0 over K. Now the homomorphism $F \to G$ extends to a specialization from $U(F)$ to $U(G)$ and this extends to a specialization of L as H-field to $U(G)$ (cf. Ch. 7 of [5]). But G is free on x, y, so the specialization must be an isomorphism, by the universality of $U(G)$, and we find that $L \cong U(G)$. This provides an embedding of $U(F)$ in $U(G)$; in particular, any full matrix over F is invertible over $U(F)$, hence also over $U(G)$ and so is full over G. Thus we have shown that the mapping $\beta_0 : F \to G$ is honest.

It remains to show that β_0 is 1-inert. Given $c \in F$, suppose that in H we have $c = ab$, $a, b \in H$. We can write $a = a_0 x^r + \dots$, $b = b_0 x^s + \dots$, where $a_0, b_0 \in F$ and dots denote terms of lower degree in x. Then $c = ab = a_0 b_0 x^{r+s} + \dots$; by uniqueness, $r + s = 0$, hence $r = s = 0$ and $a, b \in F$. This shows F to be 1-inert in H, hence the mapping β_0 is 1-inert, as we wished to show.

We can extend the scope of this result as follows.

Proposition 2.3. *Let F, G be semifirs that are K-rings, with an honest embedding $\lambda : F \to G$, and let D be a skew field containing K. Then the induced embedding $D^*_K F \to D^*_K G$ is honest.*

Proof. Our aim will be to show that $\lambda : F \to G$ induces an embedding $U(D^* F) \to U(D^* G)$. We begin by showing that

$$(6) \qquad U(D^* F) \cong U(D^* U(F)).$$

On the left we have the universal field of fractions of the semifir $D^* F$. On the right we have a field generated by the subring $D^* F$, hence a specialization of the left-hand side. It is a proper specialization precisely if some full matrix over $D^* F$ is not invertible over the right-hand side. But $D^* U(F)$ is a localization of $D^* F$, so the embedding $D^* F \to D^* U(F)$ is honest, and every full matrix over $D^* F$ is full over $D^* U(F)$, hence invertible over the right-hand side of (6). Hence the specialization is improper, i.e. (6) is an isomorphism.

It now remains to show that there is a natural embedding of $U(D^* U(F))$ in $U(D^* U(G))$. Let us write $U(F) = L$, $U(G) = M$; we have an embedding $L \to M$ and we shall show that the embedding

$$(7) \qquad D^* L \to D^* M$$

is honest; this will complete the proof.

Write $U_1 = U(D^* L)$; this is a field containing D and L and we have a natural homomorphism

$$(8) \qquad D^*_K M \to U_1{}^*_L M,$$

which reduces to the indentity on D and M. Moreover, it is an epimorphism, since the right-hand side is contained in $U(U_1^* M)$, which is generated, as a field, by D and M. Let A be a full matrix over $D^* L$; then A is invertible over $U_1 = U(D^* L)$, hence in the mapping (8) it must have come from some full matrix over $D^* M$, and this shows (7) to be an honest homomorphism, and it completes the proof.

Applying the result with $F = K < Z >, G = K < x, y >$, we obtain the first assertion of

Corollary 2.4. *The embedding $\lambda : D_K < Z > \rightarrow D_K < x, y >$ induced by the homomorhism of Th. 2.2 is honest and 1-inert.*

Now 1-inertia follows essentially as in Th. 2.2. If $c \in D_K < Z >$ satisfies $c\lambda = ab$, write $a = a_0 + a_1 + \ldots + a_r$, $b = b_0 + b_1 + \ldots + b_s$ where a_i, b_i are the terms of degree i in x, when these elements are expressed in terms of $x, z_i\lambda(= z_i\beta)$. Then $c = a_0b_0 + \ldots + a_rb_s$ and we have a contradiction, unless $r = s = 0$; but this leads to a factorization of c in $D_K < Z >$, and it proves λ to be 1-inert.

3. An Example

As an example of a full matrix over $D_K < Z >$ which is also used elsewhere (cf. [6]) we consider the following $n \times n$ matrix $C = (c_{ij})$ suggested by G.M. Bergmann (for use in [6]):

(1) $$c_{ij} = z_{n+j} d z_i, \quad \text{where} \quad d \in D,\ d \neq 0.$$

Our object is to show that this matrix C is full. Let us define, for any $m \times n$ matrix , its *inner rank* or simply *rank* rkA, as the least number r such that A can be written in the form

$$A = PQ, \quad \text{where} \quad P \quad m \times r \quad \text{and} \quad Q \quad \text{is} \quad r \times n.$$

We also recall from [5], p.253 the law of nullity: If $UV = 0$, where U is $m \times r$ and V is $r \times n$, then $rk\ U + rk\ V \leq r$.

We shall use induction on n to prove that the matrix C given by (1) is full over $R = D_K < Z >$. If this is not so, then its inner rank r is less than n and we have

(2) $$C = PQ, \quad \text{where} \quad P \quad \text{is} \quad n \times r \quad \text{and} \quad Q \quad \text{is} \quad r \times n.$$

Let us write $Q = (Q_1, Q')$, where Q_1 is the first column of Q and similarly put $C = (C_1, C')$, so that

(3) $$C_1 = PQ_1,$$

(4) $$C' = PQ'.$$

If we omit the first row and column from C, the resulting matrix is full by the induction hypothesis; hence C' has inner rank n-1 and it follows from (4) that $r = n - 1$. Now consider the homomorphism of $D_K < Z >$ obtained by mapping $z_{n+1} \mapsto 0$ and leaving the other variables unchanged. Denoting images under this homomorphism by a bar, we have, by (1), (3), (4)

$$\bar{C}_1 = \bar{P}\bar{Q}_1 = 0, \qquad \bar{C}' = \bar{P}\bar{Q}' = C'.$$

Since C' has inner rank n-1, it follows that $\bar{P}$ has inner rank n-1. By the law of nullity

$$rk\ \bar{P} + rk\ \bar{Q}_1 \leq n - 1,$$

hence $\bar{Q}_1$ has rank 0, i.e. $\bar{Q}_1 = 0$. This means that the elements in the first column of Q lie in the ideal generated by z_{n+1}; similarly the elements of the jth column of Q lie in the ideal generated by z_{n+j} and by a symmetric argument the elements in the ith row of P lie in the ideal generated by z_i. Hence in the product PQ, in any term of degree 2, there is a factor z_i on the left of a factor z_{n+j} but in C these factors are in the opposite order, and so we have a contradiction. This proves C to be full, as claimed. Now we can apply Cor. 2.4 to deduce that under the embedding of $D_F < Z >$ into $D_K < x, y >$, C maps to a full matrix.

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